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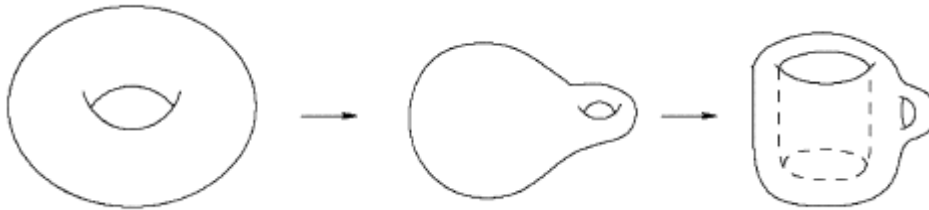
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ABSTRACT

Classification of Compact 2-manifolds

George Winslow

It is said that a topologist is a mathematician who can not tell the difference between a doughnut and a coffee cup. The surfaces of the two objects, viewed as topological spaces, are homeomorphic to each other, which is to say that they are topologically equivalent. In this thesis, we acknowledge some of the most well-known examples of surfaces: the sphere, the torus, and the projective plane. We then observe that all surfaces are, in fact, homeomorphic to either the sphere, the torus, a connected sum of tori, a projective plane, or a connected sum of projective planes. Finally, we delve into algebraic topology to determine that the aforementioned surfaces are not homeomorphic to one another, and thus we can place each surface into exactly one of these equivalence classes.



Thesis Director: Dr. Marco Aldi

Classification of Compact 2-manifolds

by

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Bachelor of Science
University of Mary Washington

Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science in the
Department of Mathematics
Virginia Commonwealth University

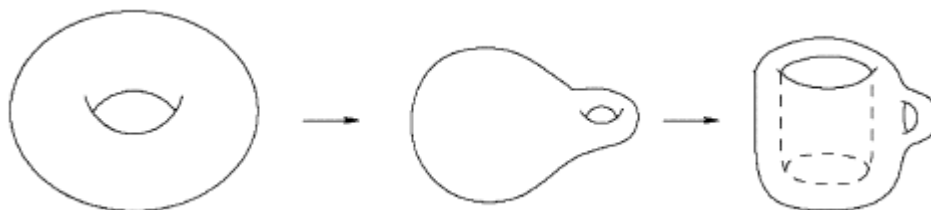
2016

DEDICATION

For Lily

ABSTRACT

It is said that a topologist is a mathematician who can not tell the difference between a doughnut and a coffee cup. The surfaces of the two objects, viewed as topological spaces, are homeomorphic to each other, which is to say that they are topologically equivalent. In this thesis, we acknowledge some of the most well-known examples of surfaces: the sphere, the torus, and the projective plane. We then observe that all surfaces are, in fact, homeomorphic to either the sphere, the torus, a connected sum of tori, a projective plane, or a connected sum of projective planes. Finally, we delve into algebraic topology to determine that the aforementioned surfaces are not homeomorphic to one another, and thus we can place each surface into exactly one of these equivalence classes.



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INTRODUCTION

In this thesis, we prove that all compact manifolds of dimension 2 are homeomorphic to the sphere, the connected sum of tori, or the connected sum of projective planes. We only assume that the reader is familiar with the basics of set theory and abstract algebra, but an understanding of point-set topology is strongly recommended.

The first chapter provides a review of point-set topology, including the basic definitions associated with topological spaces and the construction of new topological spaces from old. In particular, we rely on two properties of the quotient topology. The first is the uniqueness of the quotient topology, which states that given two quotient maps from the same space that make the same identifications, the resulting quotient spaces are homeomorphic. Most of it can be safely skipped if the reader is already comfortable with the subject matter. The second is the closed map lemma, which states that a map from a compact space to a Hausdorff space is a quotient map if it is surjective and a quotient map if it is bijective. A reader who is familiar with the subject matter can safely omit the first chapter.

We then define what we consider to be the elementary surfaces: the sphere, the torus, and the projective plane. Each will be defined rigorously and expressed in terms of quotient spaces. Next, we define the connected sum, which, intuitively, is the concept of gluing two surfaces together to create a new surface.

We employ Euclidean simplicial complexes to establish the idea that any compact surface is homeomorphic to a triangulated surface, a theorem originally proven by Tibor Rado. We follow a more modern proof by Bojan Mohar and Carsten Thomassen. We can then see that any compact surface can be expressed as a single polygon with identifications on its boundary and explore a variety of operations on such polygons that preserve the underlying surface. This polygonal presentation allows us to guarantee that any compact surface is homeomorphic to one of the three elementary surfaces or to a connected sum of elementary surfaces, following a proof by John M. Lee.

Finally, realizing that it tends to be easier to prove the existence of something than to prove that it does not exist, we use the concept of cohomotopy groups to show that there can not be any homeomorphism between any of the three main classes of surfaces, and therefore we do have three distinct homeomorphism classes.

CHAPTER 1

TOPOLOGICAL PRELIMINARIES

We begin our exploration of the homeomorphism classes of manifolds with some introductory material from general topology.

DEFINITION 1.0.1. *A topology \mathcal{T} on a set X is a collection of subsets of X with the following properties:*

- (1) *X and \emptyset are both elements of \mathcal{T} .*
- (2) *If any finite collection of subsets U_1, U_2, \dots, U_n of X are each in \mathcal{T} , then \mathcal{T} also contains the intersection $\bigcap_{k=1}^n U_k$ of all of these sets. (\mathcal{T} is closed under finite intersection.)*
- (3) *If an arbitrary collection $\{U_\alpha\}_{\alpha \in A}$ of subsets of X is contained in \mathcal{T} , then \mathcal{T} also contains the union $\bigcup_{\alpha \in A} U_\alpha$ of all of these sets.*

Note that to show the second condition, it suffices to show that intersecting two open sets yields an open set.

A set X with a topology \mathcal{T} is called a *topological space*. Subsets of X that are elements of \mathcal{T} are called *open sets*. The complement in X of an open set is called a *closed set*. If a point x is contained in an open set U , then U is called a *neighborhood* of x .

A topological space X is *connected* if there does not exist a pair of disjoint open sets U and V such that $X = U \cup V$.

If A is any subset of a topological space X , then we define the *closure* of A , denoted \overline{A} , to be the intersection of all sets containing A that are closed in X . The *interior* of A , denoted $\text{Int}(A)$, is the union of all subsets of A that are open in X . In other words, \overline{A} is the smallest closed set that contains A , and $\text{Int}(A)$ is the largest open set contained inside A . The *exterior* of A is denoted $\text{Ext}(A)$ and is defined to be the complement of \overline{A} in X . Finally, we define the *boundary* of A , denoted ∂A , to be the complement of $\text{Int}(A) \cup \text{Ext}(A)$ in X , or every point that is in neither the interior nor the exterior of A .

Let (X, \mathcal{T}_1) and (Y, \mathcal{T}_2) be topological spaces. A function $f : X \rightarrow Y$ is called an *open map* if for every open set U in X , $f(U)$ is open in Y . Similarly, $f : X \rightarrow Y$ is a *closed map* if for every closed set V in X , $f(V)$ is closed in Y . A function $f : X \rightarrow Y$ is *continuous* if for every open set U in Y , $f^{-1}(U)$, the set of all points in X that are mapped to points in U by f , is an open set in X . A *homeomorphism* is a continuous bijective map with continuous inverse. If a homeomorphism exists between two spaces, they are said to be *homeomorphic*.

Depending on how strong the conditions are on a function, these last few definitions have an interesting relationship. Let $f : X \rightarrow Y$ be a continuous bijective map. Suppose f is an open map. Then if U is an open set in X , its complement U^c is closed in X , its image $f(U)$ is open in Y , and the image of its complement (or complement of its image) $f(U^c)$ (or $(f(U))^c$) is closed in Y . The analogous statement beginning with a closed set is also true. It follows that under these conditions, all closed maps are open maps, and vice versa. Since U is open in X if and only if $f(U)$

is open in Y , f and f^{-1} are both continuous, and so f is a homeomorphism.

It is often convenient to define a topology on a set by declaring each element of a collection of subsets, called a basis, to be open and generating a topology from it.

DEFINITION 1.0.2. *A collection \mathcal{B} of subsets of a set X is a basis if it has the following properties:*

- (1) *For every $x \in X$, there is some $B \in \mathcal{B}$ such that $x \in B$. Equivalently, $\bigcup_{B \in \mathcal{B}} B = X$.*
- (2) *If B_1 and $B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.*

If \mathcal{B} is a basis in X , then we will define \mathcal{T} to be the collection of all subsets U of X such that for each $x \in U$, there is a basis element $B \in \mathcal{B}$ with $x \in B$ and $B \subseteq U$. We will verify that \mathcal{T} is a topology.

PROPOSITION 1.0.3. *The collection \mathcal{T} generated by \mathcal{B} is a topology.*

Proof. The first condition for a collection being a basis guarantees that the union of all elements in \mathcal{B} is the entire set X . The empty set is included in \mathcal{T} as well, since it is the union of no elements in \mathcal{B} .

Let U_1 and $U_2 \in \mathcal{T}$ and $x \in U_1 \cap U_2$. Choose basis elements B_1 and B_2 such that $x \in B_1 \subseteq U_1$ and $x \in B_2 \subseteq U_2$. By the definition of a basis, there is another basis element B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$. It follows that $U_1 \cap U_2$ is in \mathcal{T} , so \mathcal{T} is closed under finite intersections.

Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be a collection of sets in \mathcal{T} and $x \in \bigcup_{\alpha \in \Lambda} U_\alpha$. Then for some α , $x \in U_\alpha$. Since this U_α is in \mathcal{T} , there is a basis element B such that $x \in B \subseteq U_\alpha$. It follows that $x \in B \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha$, so $\bigcup_{\alpha \in \Lambda} U_\alpha$ is in \mathcal{T} as well. Therefore, \mathcal{T} is closed under arbitrary unions.

Having satisfied all three conditions of the definition, \mathcal{T} is a topology.

As an example, the standard topology on \mathbb{R} is that generated by the basis

$$B = \{(a, b) | a, b \in \mathbb{R}, a < b\},$$

the set of all intervals that do not include their endpoints. We will verify that this collection is a basis, thus showing that it generates a topology on \mathbb{R} .

PROPOSITION 1.0.4. *The collection B stated above is a basis.*

Proof. If x is any real number, there is an element $(x - 1, x + 1)$ of the collection that contains x , so the union of all elements is the entire space \mathbb{R} .

Suppose (a_1, b_1) and (a_2, b_2) are elements of the collection with nonempty intersection, supposing without loss of generality that $a_2 < b_1$, and let x be in their intersection. Let a be the maximum of a_1 and a_2 and b be the minimum of b_1 and b_2 . The intersection of the two intervals, then, is (a, b) , which is an element of the collection that contains x .

Since B satisfies both conditions in the definition, it is a basis, and so it generates a topology on \mathbb{R} .

1.1. NEW SPACES FROM OLD

There are several ways to construct new topological spaces based on given ones.

1.1.1. The Subspace Topology. Let (X, \mathcal{T}) be a topological space and $A \subseteq X$. There is a topology, \mathcal{T}_A , on A that is inherited from the topology on X , given by

$$\mathcal{T}_A = \{U \subseteq A : U = A \cap V \text{ for some set } V \text{ open in } X\}.$$

Now we will verify that \mathcal{T}_A , which we will call the subspace topology, is a topology.

PROPOSITION 1.1.1. *The collection \mathcal{T}_A is a topology on A .*

Proof. In X , both X and \emptyset are open sets, so $A = A \cap X$ and $\emptyset = A \cap \emptyset$ are open.

Let U_1 and U_2 be open sets in A . Then $U_1 = A \cap V_1$ and $U_2 = A \cap V_2$ for some open sets V_1 and V_2 in X . In X , then, $V_1 \cap V_2$ must be open, so $A \cap (V_1 \cap V_2) = (A \cap V_1) \cap (A \cap V_2) = U_1 \cap U_2$ is open in A .

Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be a collection of open sets in A . For each α , $U_\alpha = A \cap V_\alpha$ for some V_α open in X . The union $\bigcup_{\alpha \in \Lambda} V_\alpha$ is open in X , so $A \cap \bigcup_{\alpha \in \Lambda} V_\alpha = \bigcup_{\alpha \in \Lambda} (A \cap V_\alpha) = \bigcup_{\alpha \in \Lambda} U_\alpha$ is open in A .

Since \mathcal{T}_A has satisfied all three conditions in the definition, the collection is a topology on A .

As an example, we define the subspace $I = [0, 1]$ of \mathbb{R} . The set $[0, \frac{1}{2})$ is not open in \mathbb{R} because it can not be written as a union or finite intersection of open sets. However, it is open in I , as it can be written as $(-\frac{1}{2}, \frac{1}{2}) \cap I$, where $(-\frac{1}{2}, \frac{1}{2})$ is an open set in \mathbb{R} .

1.1.2. The Product Topology. Let X and Y be topological spaces and \mathcal{B} be the collection of all sets of the form $U \times V$, where U and V are open sets in X and Y , respectively. Then the topology on $X \times Y$ that has \mathcal{B} as a basis is called the product topology. To verify that this is a topology, we need only to show that \mathcal{B} is a basis.

PROPOSITION 1.1.2. *The collection \mathcal{B} is a basis.*

Proof. Since X and Y are open sets within themselves, the entire space $X \times Y$ is a basis element, and every point (x, y) is in $X \times Y$.

Let $U_1 \times V_1$ and $U_2 \times V_2 \in \mathcal{B}$ and $x \in (U_1 \times V_1) \cap (U_2 \times V_2)$. Since the intersections of open sets in X and Y are open, $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$ is a product of open sets and is thus a basis element.

Having met both conditions in the definition, \mathcal{B} is a basis and thus generates a topology on $X \times Y$.

As an example here, consider the Cartesian product \mathbb{R}^2 . An open set in \mathbb{R}^2 is either a set of the form $U \times V$, where U and V are open sets in \mathbb{R} , or a union or finite intersection of sets of that form.

If, for $i = 1, \dots, k$, $f_i : X_i \rightarrow Y_i$ are maps, then we will define their *product map* to be

$$f_1 \times \dots \times f_k : X_1 \times \dots \times X_k \rightarrow Y_1 \times \dots \times Y_k,$$

where

$$(f_1 \times \dots \times f_k)(x_1, \dots, x_k) = (f_1(x_1), \dots, f_k(x_k)).$$

PROPOSITION 1.1.3. *A product of continuous maps is continuous, and a product of homeomorphisms is a homeomorphism.*

Proof. Let $f_i : X_i \rightarrow Y_i$ be continuous maps for each i . It suffices to verify that the inverse images of basis open sets are open. Let $U_1 \times \dots \times U_k$ be an open set in $Y_1 \times \dots \times Y_k$. Then U_i is an open set in Y_i for each i . Then $(f_1 \times \dots \times f_k)^{-1}(U_1 \times \dots \times U_k) = f_1^{-1}(U_1) \times \dots \times f_k^{-1}(U_k)$. Since each f_i is continuous, this is a product of open sets $f_i^{-1}(U_i)$ in X_i , which is open. The same argument

applies to the inverse of a bijective product map, so if each f_i is a homeomorphism, then $f_1 \times \dots \times f_k$ is also a homeomorphism.

1.1.3. The Quotient Topology. We can also define a topology on any set if we have a topological space and a surjective map from the space to the set. Let X be a topological space, Y be a set, and $\pi : X \rightarrow Y$ be a surjective map. Define \mathcal{T}_π to be the collection of all $U \subseteq Y$ such that $\pi^{-1}(U)$ is open in X . Essentially, we define a topology on Y such that $\pi : X \rightarrow Y$ is a continuous function.

PROPOSITION 1.1.4. *The collection \mathcal{T}_π is a topology on Y .*

Proof. The collection contains Y because $\pi^{-1}(Y) = X$ since π is a surjection, and it contains the empty set because $\pi^{-1}(\emptyset) = \emptyset$.

Let U_1 and U_2 be sets in \mathcal{T}_π . By definition, $\pi^{-1}(U_1)$ and $\pi^{-1}(U_2)$ are open in X . It follows that $\pi^{-1}(U_1) \cap \pi^{-1}(U_2)$ is an open set. This intersection is the same as $\pi^{-1}(U_1 \cap U_2)$, and so $U_1 \cap U_2$ is open in Y .

Let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary collection of sets in \mathcal{T}_π . For each α , $\pi^{-1}(U_\alpha)$ is an open set in X , and so the union $\bigcup_{\alpha \in A} \pi^{-1}(U_\alpha)$ is open as well. This union is the same as $\pi^{-1}(\bigcup_{\alpha \in A} U_\alpha)$, so $\bigcup_{\alpha \in A} U_\alpha$ is an open set in Y .

Therefore, having satisfied all three conditions in the definition, \mathcal{T}_π , which we will call the *quotient topology*, is indeed a topology on Y .

We will refer to π as a *quotient map*. For any point $y \in Y$, the subset $\pi^{-1}(y) \subseteq X$ is called a *fiber* of π . A union of fibers is called a *saturated set*. Note that if a continuous surjective map $\pi : X \rightarrow Y$ takes saturated closed sets to closed sets, then it is a quotient map.

Before we move on, there are more properties of quotient maps to discuss. Let X be a topological space, Y be a set, and $\pi : X \rightarrow Y$ be a quotient map. The map π has what we will call the *universal property of quotient topologies* if for any topological space B , $f : Y \rightarrow B$ is continuous if and only if $f \circ \pi$ is continuous.

PROPOSITION 1.1.5. *Any quotient map has the universal property of quotient topologies, and any surjective map from a topological space to a set that has the universal property is a quotient map.*

Proof. If f is continuous, then $f \circ \pi$ is continuous by the nature of how π was defined. If $f \circ \pi$ is continuous, then choose an open set $U \subset B$. Then $\pi^{-1}(f^{-1}(U)) = (f \circ \pi)^{-1}(U)$ is open in X . By definition of the quotient topology, $f^{-1}(U)$ is open in Y , so f is continuous. If π is a quotient map, then, the universal property holds.

Next, we will see that if Y is given to be a topological space and $\pi : X \rightarrow Y$ is a surjective map with the universal property, then π is also a quotient map. Assuming π has the universal property, take Y itself to be the B and ι , the identity map, to be the f described in property. Since $\iota : Y \rightarrow Y$ is clearly continuous, $\iota \circ \pi = \pi$ is also continuous. Now we will show that π is a quotient map, or, equivalently, the space Y with the given topology, or Y_g , is homeomorphic to Y with the quotient topology, or Y_q . Let $\iota_{gq} : Y_g \rightarrow Y_q$ and $\iota_{qg} : Y_q \rightarrow Y_g$ be the identity functions between these two spaces and $\pi_g : X \rightarrow Y_g$ and $\pi_q : X \rightarrow Y_q$ distinguish these two similar functions. Since $\iota_{gq} \circ \pi_g = \pi_q$ is a quotient map that is defined to be continuous, ι_{gq} is continuous. Since we showed at the beginning of this paragraph that $\iota_{qg} \circ \pi_q = \pi_g$ is continuous, the universal property gives us that π_{qg} is continuous as well. These two identity functions are clearly bijective and inverses of each other, and so Y_q and Y_g are homeomorphic, making π a quotient map.

Suppose $\pi : X \rightarrow Y$ is a quotient map, B is a topological space, and $f : X \rightarrow B$ is a continuous map that is constant on the fibers of π . To rephrase this last condition, if x_1 and $x_2 \in \pi^{-1}(y)$ for some $y \in Y$, or $\pi(x_1) = \pi(x_2)$, then $f(x_1) = f(x_2)$. Then there exists a unique continuous map $\tilde{f} : Y \rightarrow B$ such that $f = \tilde{f} \circ \pi$.

Let $y \in Y$. Since π is a quotient map and is therefore surjective, there exists $x \in X$ such that $\pi(x) = y$. For all such x , let $\tilde{f}(x) = f(x)$. Since f is constant on the fibers of π , \tilde{f} is well-defined and unique, and since $f = \tilde{f} \circ \pi$ is continuous, the universal property guarantees that \tilde{f} is also continuous. A function f that fits into the hypothesis of this statement is said to *pass to the quotient*.

A result that will be very useful is that quotient spaces are uniquely determined up to homeomorphism by the identifications made by their quotient maps. Stated formally, two quotient maps $\pi_1 : X \rightarrow Y_1$ and $\pi_2 : X \rightarrow Y_2$ making the same identifications means that $\pi_1(p) = \pi_1(q)$ if and only if $\pi_2(p) = \pi_2(q)$. We will see that if π_1 and π_2 make the same identifications, then there is a unique homeomorphism $\varphi : Y_1 \rightarrow Y_2$ such that $\varphi \circ \pi_1 = \pi_2$. Both π_1 and π_2 pass uniquely to the quotient, so there are unique continuous maps $\tilde{\pi}_1 : Y_2 \rightarrow Y_1$ and $\tilde{\pi}_2 : Y_1 \rightarrow Y_2$ such that $\pi_2 = \tilde{\pi}_2 \circ \pi_1$ and $\pi_1 = \tilde{\pi}_1 \circ \pi_2$. The uniqueness of passing to the quotient gives us that $\tilde{\pi}_1 \circ \tilde{\pi}_2$ is the identity on Y_1 , and $\tilde{\pi}_2 \circ \tilde{\pi}_1 = \text{id}_{Y_2}$, the identity on Y_2 . It follows that $\tilde{\pi}_2$ is the unique homeomorphism φ we wanted.

1.2. OTHER PROPERTIES

An *open cover* of a space X is a collection $\{U_\alpha\}_{\alpha \in \Lambda}$ of open sets in X such that $\bigcup_{\alpha \in \Lambda} U_\alpha = X$. A topological space X is *compact* if every open covering of X admits a finite subcover. An important characteristic of compact spaces is that the image of a compact space under a continuous function is also compact.

THEOREM 1.2.1. *Let X and Y be topological spaces and let $f : X \rightarrow Y$ be a continuous map. If X is compact, then $f(X)$ is compact.*

Proof. Choose an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ of $f(X)$. Because f is continuous, $f^{-1}(U)$ is an open set in X for every $U \in \mathcal{U}$. Every $x \in X$ is in $f^{-1}(U)$ for some U , so $\{f^{-1}(U) : U \in \mathcal{U}\}$ is an open cover of X . Choose a finite subcover $\{f^{-1}(U_1), f^{-1}(U_2), \dots, f^{-1}(U_k)\}$ from this collection; its existence is guaranteed because X is compact. Then the collection $\{U_1, U_2, \dots, U_k\}$ covers $f(X)$, and so $f(X)$ is compact as well.

Before we move on to the next definitions, we will establish some properties of compact spaces. Every closed subset of a compact space is compact. If X is a Hausdorff space and A and B are disjoint compact subsets of A , then there exist disjoint open sets U and V in X such that $A \subseteq U$ and $B \subseteq V$. Every compact subset of a Hausdorff space is closed. Every finite product of compact spaces is compact. Every quotient of a compact space is compact.

A topological space is *second countable* if it admits a countable basis. For example, we have seen one basis for \mathbb{R} , but we will now see that the collection

$$B = \{(a, b) : a, b \in \mathbb{Q}, a < b\}$$

is also a basis.

PROPOSITION 1.2.2. *The collection \mathcal{B} mentioned above is a basis.*

Proof. If x is any real number, there is a rational number $a < x$ and another rational number $b > x$, so (a, b) is an element of the collection that contains x . It follows that the union of all elements is the entire space \mathbb{R} .

Suppose (a_1, b_1) and (a_2, b_2) are elements of the collection with nonempty intersection, supposing without loss of generality that $a_2 < b_1$, and let x be in their intersection. We may define a and b as we did when proving that the collection of all open intervals was a basis, and since both a and b are rational numbers, the intersection (a, b) is again an element of the collection that contains x .

It follows that \mathbb{R}^n has a countable basis for any natural n . Since the intersection of all basis elements with a subspace gives a basis for the subspace, any subspace of \mathbb{R}^n is second countable as well.

Second countability carries an important consequence with it. It implies a useful result that is similar to but not quite as strong as compactness.

PROPOSITION 1.2.3. *Any open cover of a second countable space admits a countable subcover.*

Proof. Let X be a second countable space and \mathcal{B} be a countable basis of X . Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ be an open cover of X . Define \mathcal{B}' to be the subcollection of all sets in \mathcal{B} that are subsets of U_α for some α . Of course, \mathcal{B}' is countable. For each $B \in \mathcal{B}'$, choose U_B from \mathcal{U} such that $B \subseteq U_B$ and define $\mathcal{U}' = \{U_B : B \in \mathcal{B}'\}$. This collection is also countable. Let $x \in X$. Then $x \in U_\alpha$. Since \mathcal{B} is a basis, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U_\alpha$, so $B \in \mathcal{B}'$. Then there is a set $U_B \in \mathcal{U}'$ such that $x \in B \subseteq U_B$. The union of all $U \in \mathcal{U}'$, then, is X , and the collection is countable.

A topological space X is *locally Euclidean of dimension n* if every point $x \in X$ has a neighborhood that is homeomorphic to \mathbb{R}^n or an open ball in \mathbb{R}^n . A seemingly convoluted but helpful result follows.

PROPOSITION 1.2.4. *If X is a second countable space, $\pi : X \rightarrow M$ is a quotient map, and M is locally Euclidean, then M is second countable.*

Proof. Let \mathcal{U} be a covering of M by Euclidean balls. Since π is a quotient map, the collection $\{\pi^{-1}(U) : U \in \mathcal{U}\}$ is made up of open sets (because of how open sets are defined in a quotient map) and covers X (because a quotient map is surjective.) Since X is second countable, we can choose \mathcal{U}' to be a countable subcover of M by Euclidean balls. Since each Euclidean ball is a subspace of \mathbb{R}^n , which is second countable, each ball has a countable basis. The countable union of all of these countable bases is itself a countable basis for M .

A topological space X is *Hausdorff* if for any distinct points x and y in X , there exist open sets U and V such that x is in U , y is in V , and the intersection of U and V is empty.

PROPOSITION 1.2.5. *The real line \mathbb{R} with the standard topology is a Hausdorff space.*

Proof. Consider two points x and y on the real line, \mathbb{R} , with $x < y$. The intervals $(x - 1, \frac{x+y}{2})$ and $(\frac{x+y}{2}, y + 1)$ are open, they contain x and y , respectively, and their intersection is empty. In \mathbb{R}^2 , two points (x_1, y_1) and (x_2, y_2) could be separated by open sets $U_1 \times V_1$ and $U_2 \times V_2$, where U_1 and U_2 separate the x coordinates and V_1 and V_2 separate the y coordinates. A similar construction gives \mathbb{R}^3 to be Hausdorff as well. Also, any subspace A of a Hausdorff space X is Hausdorff; if we choose U and V to be the disjoint open sets that separate the points in X , then $U \cap A$ and

$V \cap A$ are disjoint open sets that separate them in A .

The following result will be extremely useful in our discussion of manifolds. We will refer back to it as the Closed Map Lemma.

LEMMA 1.2.6. *Let f be a continuous map from a compact space to a Hausdorff space.*

- (1) *f is a closed map.*
- (2) *If f is surjective, it is a quotient map.*
- (3) *If f is bijective, it is a homeomorphism.*

Proof. Suppose $A \subseteq X$ is closed. Any closed subset of a compact space is compact, so A is compact. Since the continuous image of a compact set is compact, $f(A)$ is also compact. Since compact subsets of Hausdorff spaces are closed, $f(A)$ is also closed. Therefore, f is a closed map.

Note that if f is also surjective, we have satisfied all of the conditions for it to be a quotient map. Also, if f is bijective, then it is a homeomorphism.

LEMMA 1.2.7. *Let $X = A \cup B$, where A and B are closed in X . Let $f : A \rightarrow Y$ and $g : B \rightarrow Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h : X \rightarrow Y$, defined by setting $h(x) = f(x)$ if $x \in A$, and $h(x) = g(x)$ if $x \in B$.*

Proof. Let C be a closed subset of Y . By elementary set theory,

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C).$$

Since f is continuous, $f^{-1}(C)$ is closed in A and, therefore, closed in X . Similarly, $g^{-1}(C)$ is closed in B and thus also in X . Their union $h^{-1}(C)$ is closed in X .

We now have enough definitions and facts to identify the main subject of interest: the manifold.

DEFINITION 1.2.8. *An n -dimensional topological manifold, or n -manifold, is a second countable Hausdorff space that is locally Euclidean of dimension n .*

Consider as an example the set

$$\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

PROPOSITION 1.2.9. *The one-dimensional sphere \mathbb{S}^1 is a 1-manifold.*

Proof. As a subspace of \mathbb{R}^2 , \mathbb{S}^1 is a second countable Hausdorff space. To show that it is locally Euclidean, take a point (x, y) from the space. If y is not zero, then choose an open disk in \mathbb{R}^2 that is contained entirely either above the x -axis or below, depending on which half contains (x, y) . The projection map $\pi_1(x, y) = x$ between the intersection of that disk and \mathbb{S}^1 to the interval on the x -axis that is directly above or below it is a homeomorphism. If y is zero, then similarly choose an open disk in \mathbb{R}^2 that is contained either to the left or right of the y -axis, depending on where (x, y) is and use the projection map $\pi_2(x, y) = y$. Since every point has a neighborhood homeomorphic to an open disk in \mathbb{R} , \mathbb{S}^1 is locally Euclidean and thus is a 1-dimensional manifold, or 1-manifold.

We conclude this chapter with one more result concerning manifolds as product spaces.

PROPOSITION 1.2.10. *If M_1, \dots, M_k are manifolds of dimensions n_1, \dots, n_k , respectively, then the product space $M = M_1 \times \dots \times M_k$ is a manifold of dimension $n_1 + \dots + n_k$.*

Proof. Since each M_i is a manifold, each is second countable, Hausdorff, and locally Euclidean of dimension n_i . We will show each of these properties, one by one.

Let $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$ be distinct points in M . For at least one i , x_i and y_i are distinct points in M_i . Since M_i is a manifold and thus a Hausdorff space, there exist disjoint open sets U and V in M_i that contain x_i and y_i , respectively. Then $M_1 \times \dots \times U \times \dots \times M_k$ and $M_1 \times \dots \times V \times \dots \times M_k$ are each disjoint open sets that contain x and y , respectively, so M is a Hausdorff space.

Let \mathcal{B}_i be a countable basis for M_i for each i . Consider the collection

$$\mathcal{B} = \{B_1 \times \dots \times B_k : B_i \in \mathcal{B}_i\}.$$

If $x = (x_1, \dots, x_k)$ is in the product of manifolds, then each x_i is in M_i and is thus in a basis element B_i of \mathcal{B}_i . It follows that x is in $B_1 \times \dots \times B_k$, so the collection covers M . Suppose there are two basis elements $B_{1,1} \times \dots \times B_{1,k}$ and $B_{2,1} \times \dots \times B_{2,k}$ that both contain a point x . Then for each i , $x_i \in B_{1,i} \cap B_{2,i}$, and so there exists another basis element $B_{3,i}$ such that $x_i \in B_{3,i} \subseteq B_{1,i} \cap B_{2,i}$. It follows that $x \in B_{3,1} \times \dots \times B_{3,k} \subseteq (B_{1,1} \times \dots \times B_{1,k}) \cap (B_{2,1} \times \dots \times B_{2,k})$. Being a finite product of countable sets, \mathcal{B} is countable, and since it is a basis, M is a second countable space. (Note that while [Lee] includes the stipulation of second countability in the definition of a manifold, we are concerned with compact manifolds, which can all be shown to be second countable.)

Each x_i has a neighborhood U_i in M_i that is homeomorphic by a homeomorphism f_i to an open set in \mathbb{R}^{n_i} . By a result proved in our discussion of product spaces, the product map $f_1 \times \dots \times f_k$ is a homeomorphism from $U_1 \times \dots \times U_k$, a neighborhood of x , to an open set in $\mathbb{R}^{n_1+\dots+n_k}$. The product space M is locally Euclidean of dimension $n_1 + \dots + n_k$, and thus it is an $(n_1 + \dots + n_k)$ -manifold.

CHAPTER 2

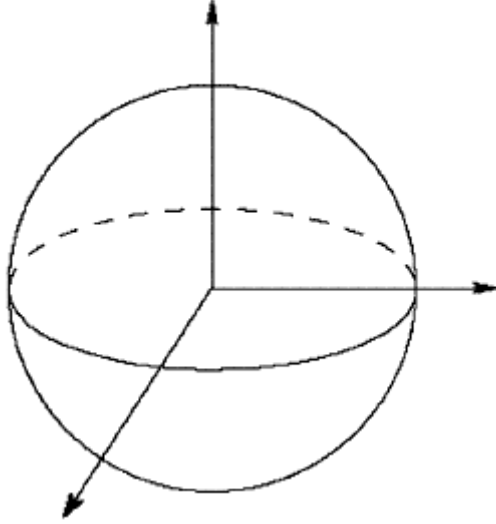
BASIC SURFACES

We are concerned with 2-dimensional compact connected manifolds, or surfaces. The three most important surfaces we will discuss are the sphere, the torus, and the projective plane. In the following sections, we will rigorously define each of these three surfaces and examine various quotient spaces involving the unit interval product $I \times I$ to which each is homeomorphic, which will aid in visualizing them. To show that the quotient spaces are homeomorphic to a surface in question, we will define a quotient map from the quotient space to the surface that makes the same identifications as an equivalence relation defined on the space.

2.1. \mathbb{S}^2 , THE SPHERE

The unit n -sphere, denoted \mathbb{S}^n , is the set of all points in Euclidean n -space that are exactly one unit away from the origin. The 2-dimensional sphere is

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$



To verify that \mathbb{S}^2 is a manifold, consider \mathbb{S}^2 as a subspace of \mathbb{R}^3 . Define $\sigma : \mathbb{S}^2 \setminus \{(0, 0, 1)\} \rightarrow \mathbb{R}^2$ by

$$\sigma(x, y, z) = \frac{(x, y)}{1 - z}.$$

This function, called a stereographic projection, sends a point (x, y, z) on the sphere to the point on the xy -plane where a line going through $(0, 0, 1)$ and (x, y, z) intersects the plane. Its inverse is

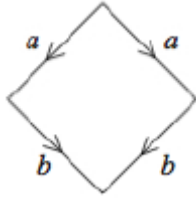
$$\sigma^{-1}(x, y) = \frac{(2x, 2y, x^2 + y^2 - 1)}{x^2 + y^2 + 1}.$$

Since both σ and its inverse are continuous bijections, $\mathbb{S}^2 \setminus \{(0, 0, 1)\}$ is homeomorphic to \mathbb{R}^2 . The set $\mathbb{S}^2 \setminus \{(0, 0, 1)\}$ is an open set containing all but one point on the sphere, and so every point except for possibly $(0, 0, 1)$ has a Euclidean neighborhood. To resolve the issue for the last point, let $U = \{(x, y, z) \in \mathbb{R}^3 : z > 0\} \cap \mathbb{S}^2$, the open top half of the sphere, and V be the open ball of radius 1 around the origin in \mathbb{R}^2 . Define $\pi : U \rightarrow V$ by $\pi(x, y, z) = (x, y)$. Then $\pi^{-1} : V \rightarrow U$ may be given by $\pi^{-1}(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$. It follows that U is a neighborhood of $(0, 0, 1)$

that is homeomorphic to an open ball in \mathbb{R}^2 , and so \mathbb{S}^2 is locally Euclidean in 2-space. Its Hausdorff property and second countability follow from its being a subspace of \mathbb{R}^3 .

PROPOSITION 2.1.1. *The sphere \mathbb{S}^2 is homeomorphic to the following quotient spaces:*

- (1) *The closed disk $\overline{\mathbb{B}^2} \subset \mathbb{R}^2$ modulo the equivalence relation generated by $(x, y) \sim (-x, y)$ for $x \in \partial\mathbb{B}^2$.*
- (2) *The square $I \times I$ modulo the equivalence relation generated by $(0, t) \sim (t, 0)$ and $(t, 1) \sim (1, t)$ for $0 \leq t \leq 1$.*



Proof. Define $\pi : \overline{\mathbb{B}^2} \rightarrow \mathbb{S}^2$ by

$$\pi(x, y) = \begin{cases} \left(-\sqrt{1-y^2} \cos \frac{\pi x}{\sqrt{1-y^2}}, -\sqrt{1-y^2} \sin \frac{\pi x}{\sqrt{1-y^2}}, y \right) & y \neq \pm 1; \\ (0, 0, y) & y = \pm 1. \end{cases}$$

By the closed map lemma, this is a quotient map that, due to the periodicity of the sine and cosine functions, makes the same identifications as the equivalence relation $(x, y) \sim (-x, y)$ on the boundary of the disk. The uniqueness of quotient maps guarantees that the two structures are homeomorphic.

For the second part, define a homeomorphism α from $I \times I$ to \mathbb{R}^2 by $\alpha(x, y) = (2x - 1, 2y - 1)$. This function simply doubles the area of the square and translates it so that its center is at the origin. Next, define $\beta : \alpha(I \times I) \rightarrow \overline{\mathbb{B}^2}$ to be a homeomorphism that sends each line segment between the origin and a point on the boundary of the square to a segment between the center of the closed disk and its

boundary. All that remains to ensure that this map makes the same identifications as the previous one is to define one last homeomorphism γ from the closed disk to itself, rotating it clockwise by $\frac{\pi}{4}$ radians. The composition of $\pi \circ \gamma \circ \beta \circ \alpha$ completes the quotient map from $I \times I$ to \mathbb{S}^2 , which makes the same identifications as the equivalence relation on $I \times I$ mentioned above.

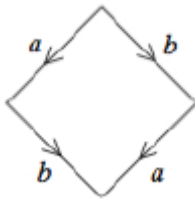
2.2. \mathbb{T}^2 , THE TORUS

The n -torus, denoted \mathbb{T}^n , is the product manifold obtained from the product of n copies of \mathbb{S}^1 . In 2 dimensions, the torus is

$$\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1.$$

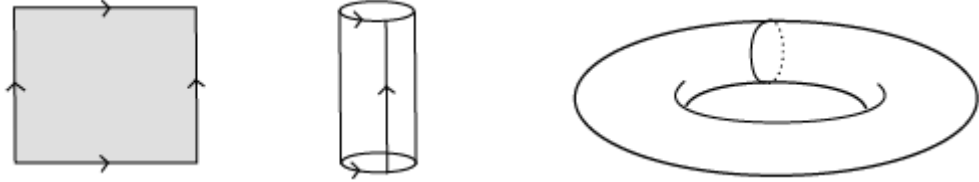
As we saw at the end of the last chapter, as the product of two 1-manifolds, the torus is a 2-manifold. Note that we can express the torus as a set of ordered quadruples (x_1, x_2, x_3, x_4) , where $x_1^2 + x_2^2 = x_3^2 + x_4^2 = 1$.

PROPOSITION 2.2.1. *The torus \mathbb{T}^2 is homeomorphic to $I \times I$ modulo the equivalence relation generated by $(t, 0) \sim (t, 1)$ and $(0, t) \sim (1, t)$ for $0 \leq t \leq 1$.*



Proof. Define $f : I \times I \rightarrow \mathbb{T}^2$ by $f(x, y) = (\cos 2\pi x, \sin 2\pi x, \cos 2\pi y, \sin 2\pi y)$. The closed map lemma again verifies that this is a quotient map, and it makes the same identifications as the equivalence relation on $I \times I$. It follows that the quotient space is homeomorphic to the torus.

Having seen this argument, we can visualize the torus as the result of gluing the two sides of $I \times I$ to make a cylinder, and then gluing the top and bottom together to yield a donut-like surface.



2.3. \mathbb{P}^2 , THE PROJECTIVE PLANE

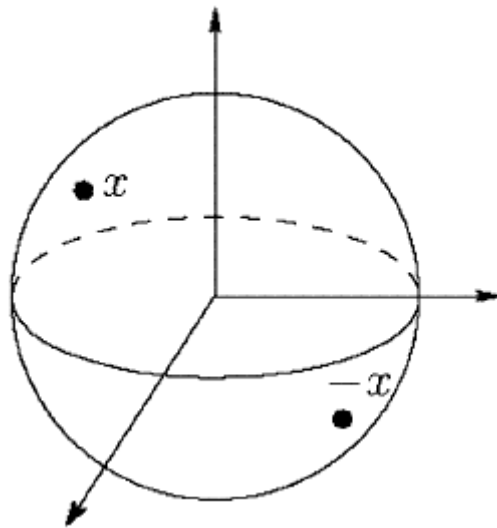
The real projective space of dimension n , denoted \mathbb{P}^n , is the set of all lines through the origin in $(n+1)$ -space. We are concerned with \mathbb{P}^2 , the set of all lines through the origin in \mathbb{R}^3 . Of course, in order to discuss \mathbb{P}^2 , there must be a topology on it. Define $\pi : \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{P}^2$, naturally, by setting $\pi(x, y, z)$ to be the line that goes through both (x, y, z) and the origin. This function is clearly well-defined and surjective, so we can use it to define the quotient topology \mathcal{T}_π on \mathbb{P}^2 .

Essentially, an open set looks like a cluster of lines, without the boundary included, going through the origin. If x_1 and x_2 are two such lines, we need only choose a nonzero point on each line and choose disjoint open sets U and V around containing those two points in \mathbb{R}^3 to see that \mathbb{P}^2 is a Hausdorff space. The image of U and V under π are disjoint open sets in \mathbb{P}^2 that contain the two lines. Since \mathbb{R}^3 is second countable, we may simply take the image of a countable basis of \mathbb{R}^3 and use its image under π as a countable basis of \mathbb{P}^2 . To see intuitively that it is locally Euclidean of dimension 2, choose a line and a neighborhood around it, and visualize a cross section of that cluster that is parallel to either the xy -plane, the yz -plane, or the xz -plane.

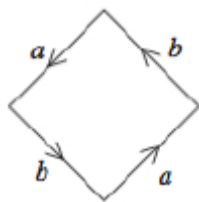
The result is a flat disk without boundary in which every point represents one of the lines from the original open set and yields an obvious homeomorphism to a disk in \mathbb{R}^2 .

PROPOSITION 2.3.1. *The projective plane \mathbb{P}^2 is homeomorphic to each of the following quotient spaces:*

- (1) *The sphere \mathbb{S}^2 modulo the equivalence relation $x \sim -x$ for each $x \in \mathbb{S}^2$.*



- (2) *The closed disk $\overline{\mathbb{B}^2}$ modulo the relation $(x, y) \sim (-x, -y)$ for each $(x, y) \in \partial\mathbb{B}^2$.*
- (3) *The square $I \times I$ modulo the equivalence relation $(t, 0) \sim (1-t, 1), (0, 1-t) \sim (1, t)$ for $0 \leq t \leq 1$.*



Proof. Let \mathbb{S}^2 / \sim denote the quotient space resulting from the equivalence relation in the first part and let $\varphi : \mathbb{S}^2 \rightarrow \mathbb{S}^2 / \sim$ be the quotient map. The first part is easy to see. A line from a point on \mathbb{S}^2 through the origin is the same as the line from its antipodal point through the origin, so the inclusion map of \mathbb{S}^2 into $\mathbb{R}^3 \setminus \{0\}$ composed with π , the map we used to define the topology on \mathbb{P}^2 , clearly makes the same identifications as the equivalence relation, and, as in our previous examples, it is a quotient map by the closed map lemma.

For the second part, compose φ with the function $\theta : \overline{\mathbb{B}^2} \rightarrow \mathbb{S}^2$ given by $\theta(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$. The function θ curves the disk into the upper hemisphere, and the composition with φ identifies a point on the upper hemisphere with its antipodal point. Again, the closed map lemma guarantees that it is a quotient map, and it makes the same identifications as our equivalence relation, so the quotient space is homeomorphic to \mathbb{P}^2 .

With these two homeomorphisms shown, we can use the composition $\pi \circ \beta \circ \alpha$, the same functions as in the argument for \mathbb{S}^2 , only without the rotation γ , to see that \mathbb{P}^2 is homeomorphic to $I \times I$ with the given equivalence relation.

Depending on your perspective, the equivalence relation on $I \times I$ yields a space that is homeomorphic to \mathbb{P}^2 may not be any more helpful in visualizing what the projective plane looks like. Intuitively, this construction means we begin with the square, twist it, and glue it together along the sides so that the bottom left and top right corners and the top left and bottom right corners match, then twist again before gluing the top and bottom together. This procedure obviously can not be performed in a physical sense, but as mentioned at the beginning of the chapter, having this

presentation of \mathbb{P}^2 will be helpful to us.

The Möbius strip, the closed disk, and the cylinder are conspicuously absent because they are not manifolds. The points on the boundaries of these objects do not have neighborhoods that are homeomorphic to open disks in \mathbb{R}^2 , and so they fail to be locally Euclidean.

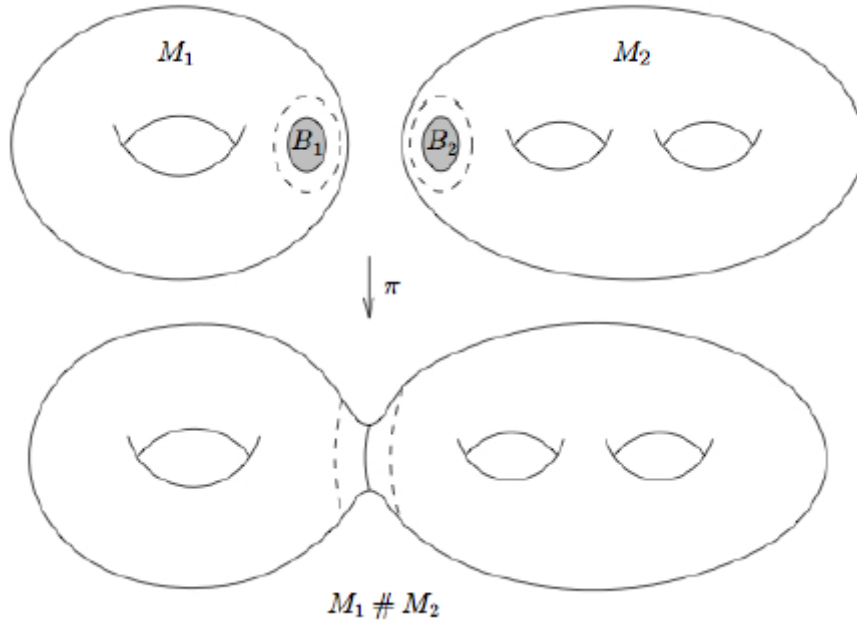
2.4. CONNECTED SUMS

New manifolds can be created from the basic three, or from any existing manifolds, by joining them. The joining process can be visualized by imagining the act of cutting a small circle out of each surface and then gluing the two surfaces together along the two boundaries. For example, gluing two tori together would yield a two-holed torus. This operation on manifolds is called the *connected sum*.

DEFINITION 2.4.1. *Suppose X and Y are topological spaces, A is a closed subset of Y , and $f : A \rightarrow X$ is a continuous map. We define an equivalence relation \sim on the disjoint union $X \amalg Y$ such that $a \sim f(a)$ for all $a \in A$. The resulting quotient $(X \amalg Y)/\sim$, denoted $X \cup_f Y$, is an adjunction space.*

A connected sum is a special case of an adjunction space given a particular choice of the closed set A .

DEFINITION 2.4.2. *Given surfaces M_1 and M_2 , let $B_i \subset M_i$ be Euclidean balls. The two boundaries are both homeomorphic to \mathbb{S}^1 , so they are homeomorphic to each other. Choose a homeomorphism $\sigma : \partial B_1 \rightarrow \partial B_2$. Define $M'_i = M_i \setminus B_i$. The connected sum of M_1 and M_2 , denoted $M_1 \# M_2$, is the adjunction space $M'_1 \cup_\sigma M'_2$.*



PROPOSITION 2.4.3. *Given two connected surfaces M_1 and M_2 , the connected sum $M_1 \# M_2$ is a connected surface.*

Proof. For $M_1 \# M_2$ to be a surface, it must be a locally Euclidean Hausdorff space. Showing the Hausdorff property is fairly trivial; any two distinct points can easily be separated by choosing small enough open balls. We will begin by showing that it is locally Euclidean. Let $\pi : M'_1 \amalg M'_2 \rightarrow M_1 \# M_2$ be the quotient map mentioned in the definition.

There are two types of points in $M_1 \# M_2$, classified by the locations of their preimages: those whose preimages are in the interior of either M'_1 or M'_2 , and those points whose preimages are in both $\partial M'_1$ and $\partial M'_2$. The points whose preimages are in the interior clearly have Euclidean neighborhoods; we need only choose a neighborhood strictly

contained within their respective M'_i that is homeomorphic to \mathbb{R}^2 .

For the second type of point, let p_1 be the preimage in M'_1 and p_2 be the preimage in M'_2 that is identified with p_1 . There exist disjoint neighborhoods U_1 of p_1 and U_2 for p_2 such that $M'_1 \cap U_1$ and $M'_2 \cap U_2$ are disjoint half-disks. Let $V_1 = M'_1 \cap U_1$ and $V_2 = M'_2 \cap U_2$. We can construct affine homeomorphisms α_1 and α_2 such that α_1 maps V_1 to a half-disk on the upper half plane, α_2 maps V_2 to a half disk on the lower half plane, p_1 and p_2 are both mapped to the origin, and boundary identifications are respected. If necessary, we can shrink V_1 and V_2 until $V_1 \cup V_2$ is a saturated open set in the plane; i.e., for every point in $V_1 \cap \partial P$, the corresponding boundary point is in V_2 and vice versa. We may now define another homeomorphism $\alpha : V_1 \cup V_2 \rightarrow \mathbb{R}^2$ such that $\alpha(q) = \alpha_1(q)$ for all $q \in V_1$ and $\alpha(q) = \alpha_2(q)$ for all $q \in V_2$. Define an equivalence relation \sim so that for any edge points r_1 and r_2 in V_1 and V_2 , respectively, $r_1 \sim r_2$ when $\alpha(r_1) = \alpha(r_2)$. Since $\pi|_{V_1 \cup V_2}$ is a quotient map, and α is a quotient map onto an open ball centered at the origin and making the same identifications as π . By the uniqueness of the quotient map, the quotient spaces are homeomorphic, so every point on the now shared boundary has a Euclidean neighborhood.

To see that $M_1 \# M_2$ is connected, note that $M_1 \# M_2$ is the union of two connected sets $\pi(M'_1)$ and $\pi(M'_2)$, where $\pi(M'_1) \cap \pi(M'_2) \neq \emptyset$.

CHAPTER 3

TRIANGULATION OF SURFACES

3.1. SIMPLICIAL COMPLEXES

In the previous chapter, we saw that our three basic surfaces can be constructed by making identifications on the edges of the unit square. Now, we will expand upon that idea by showing that any surface can be constructed from a polygon with identifications on the edges. In order to verify this claim, we will observe surfaces in the context of Euclidean simplicial complexes.

DEFINITION 3.1.1. *Let points $v_0, v_1, \dots, v_k \in \mathbb{R}^n$ be such that $\{v_1 - v_0, v_2 - v_0, \dots, v_k - v_0\}$ are linearly independent. The simplex σ spanned by them is the set of points*

$$\{x \in \mathbb{R}^n \mid x = \sum_{i=0}^k t_i v_i \text{ such that } 0 \leq t_i \leq 1 \text{ and } \sum_{i=0}^k t_i = 1\}$$

with the subspace topology inherited from \mathbb{R}^n . Each v_i is a vertex of σ . The simplex σ spanned by vertices $\{v_0, v_1, \dots, v_k\}$ is denoted by $\langle v_0, v_1, \dots, v_k \rangle$.

The *dimension* of simplex σ is one fewer than the number of vertices. A simplex of dimension k is called a k -simplex. For example, a 0-simplex has one vertex and is represented as a single point. A 1-simplex has two vertices and is represented by the line segment between them in \mathbb{R}^2 . A 2-simplex has three vertices, and as they are linearly independent, the three vertices will be the three vertices of a triangle whose boundary and interior constitute the simplex. A 3-simplex would have four linearly

independent vertices and so would be a tetrahedron. For higher k , the simplex can not be realized in \mathbb{R}^3 .



Let σ be a simplex with vertices $\{v_0, v_1, \dots, v_k\}$. We refer to the simplex spanned by any non-empty subset of $\{v_0, v_1, \dots, v_k\}$ as a *face* of σ . The simplex spanned by a proper subset of $\{v_0, v_1, \dots, v_k\}$ is called a *proper face* of σ . The faces that are of dimension $(k - 1)$, where k is the dimension of σ , are called *boundary faces*.

The union of the boundary faces of a simplex σ is called the *boundary* of σ and is denoted $\partial\sigma$. The *interior* of a simplex σ , denoted $\text{Int } \sigma$, is the simplex minus its boundary.

A collection of simplices can be combined to create a simplicial complex.

DEFINITION 3.1.2. A simplicial complex is a collection K of simplices in \mathbb{R}^n satisfying each of the following conditions:

- (1) If $\sigma \in K$, then every face of σ is also in K .
- (2) Let σ_1 and $\sigma_2 \in K$. If $\sigma_1 \cap \sigma_2$ is nonempty, then it must be a face of both σ_1 and σ_2 .
- (3) Every point in a simplex $\sigma \in K$ has a neighborhood that intersects finitely many simplices of K .

The *dimension* of a simplicial complex K is the maximum dimension of any simplex in K . We will be mainly concerned with 2-dimensional simplicial complexes, in which the second condition above means that when simplices intersect, they do so at either vertices or edges.

By $|K|$, we mean the union of all simplices in K with the subspace topology inherited from \mathbb{R}^n .

DEFINITION 3.1.3. Let K be a simplicial complex. For any non-negative integer k , we denote by $K^{(k)}$ the subset of K consisting of all simplices of dimension less than or equal to k . This subset, or subcomplex, is called the k -skeleton of K .

PROPOSITION 3.1.4. Let $\sigma = \langle v_0, v_1, \dots, v_k \rangle$ be a k -simplex in \mathbb{R}^n . Given $k + 1$ points $w_0, w_1, \dots, w_k \in \mathbb{R}^m$, there exists a unique map $f : \sigma \rightarrow \mathbb{R}^m$ that is the restriction of an affine map that maps v_i to w_i for each i .

Proof. We may assume without loss of generality that both v_0 and w_0 are equal to the zero vector in their respective spaces. It follows from the definition of a k -simplex that $\{v_1, v_2, \dots, v_k\}$ are linearly independent. Let $f : \sigma \rightarrow \mathbb{R}^m$ be the restriction of any linear map such that $v_i \mapsto w_i$ for each i . Because we chose a linear map, we can say that, for each $v \in \sigma$,

$$f(v) = f\left(\sum_{i=0}^k t_i v_i\right) = \sum_{i=0}^k t_i f(v_i),$$

such that $0 \leq t_i \leq 1$ and $\sum_{i=0}^k t_i = 1$, the uniqueness of f is guaranteed.

DEFINITION 3.1.5. Let K and L be simplicial complexes. A simplicial map is a continuous map $f : |K| \rightarrow |L|$ such that the restriction of f to each simplex of K maps to some simplex in L via an affine map.

The restriction of a simplicial map f to $K^{(0)}$, which is the set of vertices of K , yields a map $f_0 : K^{(0)} \rightarrow L^{(0)}$ called the *vertex map* of f .

A simplicial map that is also a homeomorphism in accordance with the topological structure on $|K|$ and $|L|$ is called a *simplicial isomorphism*.

LEMMA 3.1.6. Let K and L be simplicial complexes. Suppose $f_0 : K^{(0)} \rightarrow L^{(0)}$ is any map such that if $\{v_0, v_1, \dots, v_k\}$ are vertices of a simplex of K , then $\{f_0(v_0), f_0(v_1), \dots, f_0(v_k)\}$ are vertices of a simplex of L . Then there exists a unique simplicial map $f : |K| \rightarrow |L|$ such that f_0 is the vertex map of f .

Proof. Assuming the conditions of the lemma, choose $f : |K| \rightarrow |L|$ such that the restriction of f to any simplex $\sigma = \langle v_0, v_1, \dots, v_k \rangle$ maps the vertices of σ to the vertices $\{f_0(v_0), f_0(v_1), \dots, f_0(v_k)\}$ of some simplex in L in accordance with f_0 . Clearly, the simplex spanned by $\{f_0(v_0), f_0(v_1), \dots, f_0(v_k)\}$ is the convex hull $\langle f_0(v_0), f_0(v_1), \dots, f_0(v_k) \rangle$, so f is a simplicial map. Its uniqueness is proven by the fact that for each $v \in \sigma$,

$$f(v) = f\left(\sum_{i=0}^k t_i v_i\right) = \sum_{i=0}^k t_i f(v_i) = \sum_{i=0}^k t_i f_0(v_i),$$

such that $0 \leq t_i \leq 1$ and $\sum_{i=0}^k t_i = 1$.

LEMMA 3.1.7. *Let K and L be simplicial complexes with f and f_0 as stated above.*

If both of the following statements are true, then f is a simplicial isomorphism:

- (1) *The vertex map f_0 is bijective,*
- (2) *$\{v_0, v_1, \dots, v_k\}$ are vertices of a simplex of K if and only if $\{f_0(v_0), f_0(v_1), \dots, f_0(v_k)\}$ are vertices of a simplex of L .*

Proof. We have already established that f is a simplicial map. It remains, then, to prove that f is a homeomorphism from $|K|$ to $|L|$. Since f_0 is bijective, K and L have the same number of vertices. The set $\{v_0, v_1, \dots, v_k\}$ is a set of vertices of some simplex in K if and only if $\{f_0(v_0), f_0(v_1), \dots, f_0(v_k)\} = \{w_0, w_1, \dots, w_k\}$ are distinct vertices of a simplex in L . It follows that $\langle v_0, v_1, \dots, v_k \rangle$ and $\langle w_0, w_1, \dots, w_k \rangle$ are k -simplices in K and L , respectively, so σ is a k -simplex in K if and only if $f_0(\sigma)$, the convex hull of the set of $f_0(v_i)$ for all i , is a k -simplex in L . We have shown that $|K|$ and $|L|$ are homeomorphic, and therefore f is a simplicial isomorphism.

We can refer to a topological space that is homeomorphic to a simplicial complex as a *polyhedron*. The homeomorphism between a topological space and a simplicial complex is called a *triangulation*.

Having established the terminology for simplicial complexes and triangulations, we will now take an aside to prove that every surface is homeomorphic to a triangulated surface.

3.2. GRAPH THEORY PRELIMINARIES

Our main result requires some background in graph theory, which will be presented in this chapter.

DEFINITION 3.2.1. *A graph G is a triple (V, E, st) . The first element, V , is a set of points called vertices, *sg.* vertex. The set E consists of edges, which are line segments or arcs that connect vertices. Finally, $st : E \rightarrow V \times V$ is a function that assigns to each edge e in E a pair of vertices $\{u, v\}$ in V .*

For our purposes, we may assume that V and E are finite sets, that u and v are distinct, and that no two distinct edges map to the same pair of vertices. If $st(e) = \{u, v\}$, we say that vertices u and v are *adjacent* and edge e is *incident* to vertices u and v .

A graph $H = (V_H, E_H, st_H)$ is a *subgraph* of $G = (V, E, st)$ if $V_H \subset V$, $E_H \subset E$, and st_H is the restriction of st to E_H .

Two graphs $G_1 = (V_1, E_1, st_1)$ and $G_2 = (V_2, E_2, st_2)$ are *isomorphic* if there is a pair of bijections $f_V : V_1 \rightarrow V_2$ and $f_E : E_1 \rightarrow E_2$ such that the incidence of edges is preserved; i.e., for each edge $e \in E_1$, if $st_1(e) = \{u, v\}$, then $st_2(f_E(e)) = \{f_V(u), f_V(v)\}$. The pair of bijections are collectively called an *isomorphism*.

A *chain* in G is a sequence (v_0, v_1, \dots, v_n) , with $n \geq 1$, where each $v_i \in V$ and for each e_i such that $st(e_i) = \{v_{i-1}, v_i\}$, $e_i \in E$. A *path* is a chain in which all vertices are distinct. A *cycle* is a path in which $n \geq 2$ and $v_0 = v_n$. A graph G is *connected* if for any two vertices u and v in V , there is a path in G containing both.

We can use the cycles in a graph G to partition the set E of edges and define an equivalence relation \sim . For any two edges $e_1, e_2 \in E$, $e_1 \sim e_2$ if there is a cycle in G containing both e_1 and e_2 . For an edge e that is not contained in any cycles, we will add the provision that $e \sim e$ for all $e \in E$. Under the relation \sim , each equivalence class of edges along with incident vertices is called a *block*. A vertex with no incident edges, called an *isolated vertex*, is also a block. An edge that is not a part of any cycle and thus is in its own block is called a *cutedge* or *bridge*. A vertex that is contained in more than one block is called a *cutvertex*.

For any vertex v , we define $G - v$ to be the subgraph $(V - \{v\}, E', st')$, where E' is the subset of E remaining after removing all edges incident with vertex v and st' is the restriction of st to E' . A graph G is *2-connected* if it has at least 3 vertices and $G - v$ is connected for all $v \in V$. Clearly, 2-connectedness is a stronger condition than connectedness; any 2-connected graph is obviously also connected.

PROPOSITION 3.2.2. *Any two distinct blocks in a graph have at most one vertex in common, and if they do have a vertex in common, then it is a cutvertex.*

Proof. Let $G = (V, E, st)$ be a graph made up of at least two blocks $B_1 = (V_1, E_1, st_1)$ and $B_2 = (V_2, E_2, st_2)$, and suppose v_1 and v_2 are both in contained in V_1 and V_2 . As both blocks contain at least two vertices, they are not isolated, and thus their edge sets are nonempty. Therefore, there is a cycle in B_1 containing both v_1 and v_2 , which implies that there is a path in B_1 from v_1 to v_2 . Similarly, there is a

path in B_2 from v_2 to v_1 . Concatenating these two paths will yield a cycle containing both v_1 and v_2 that has edges in both B_1 and B_2 , which means that the edges in B_1 are actually in the same equivalence class as those in B_2 and thus they are parts of the same block. As this fact contradicts our assumption that the two blocks were distinct, it is not possible for two distinct blocks to have more than one vertex in common. By definition, that vertex is a cutvertex.

PROPOSITION 3.2.3. *Let G be a graph.*

- (a) *An edge e is a cutedge if and only if the vertices incident to e belong to distinct connected components of $G - e$.*
- (b) *A vertex v is a cutvertex if and only if $G - v$ contains more components than G .*
- (c) *If u and v are two vertices in a block B containing at least 2 edges, then B contains a cycle containing u and v .*
- (d) *Two blocks of G have at most one vertex in common, and such a vertex is a cutvertex of G .*
- (e) *If B_1, B_2, \dots, B_k ($k \geq 3$) are distinct blocks in G such that $B_i \cap B_{i+1} \neq \emptyset$ for $i = 1, 2, \dots, k-1$, and $B_1 \cap B_2, B_2 \cap B_3, \dots, B_{k-1} \cap B_k$ are distinct, then $B_1 \cap B_k = \emptyset$.*

Proof. These proofs are all trivial or similar enough to those stated earlier that they need not be included. They can also be found in [Mohar and Thomassen].

PROPOSITION 3.2.4. *Let G be a connected graph with at least three vertices. The following are equivalent.*

- (a) *G is 2-connected*
- (b) *any 2 vertices of G are on a common cycle*

- (c) any 2 edges of G are on a common cycle
- (d) G has no cutvertices
- (e) For every vertex v of G , the graph $G - v$ is connected
- (f) G has only one block.

Proof. This proof is also very straightforward and can be found in [Mohar and Thomassen].

PROPOSITION 3.2.5. *If G is a 2-connected graph, then it can be obtained from a cycle of length at least three by successively adding a path having only its ends in common with the current graph.*

Proof. Let $G = (V, E, st)$ be the desired 2-connected graph and $H = (V_H, E_H, st_H)$ the 2-connected subgraph. There is nothing to say if $H = G$, so we assume that $H \neq G$. Then since G is connected, there must be some edge $e \in E \setminus E_H$ that is incident to vertices u and v with $u \in V_H$ and $v \in (V \setminus V_H)$. Since G is 2-connected, $G - u$ is still connected. Let Q be the shortest path in $G - u$ from v to an arbitrary vertex w in H . Due to the nature of our choice of Q , all edges in Q must be in $E \setminus E_H$. Let P be the path consisting of the edge between u and v followed by the full path Q . The path P has only its end vertices, u and w , in H , and none of its edges in H . Adding P to H yields a new 2-connected graph. Since every edge and vertex that we add is part of G , repeating the process will eventually yield G .

A *simple polygonal arc* in \mathbb{R}^2 is the union of a finite number of line segments S_1, S_2, \dots, S_n with S_i intersecting only S_{i+1} at only one endpoint, except that S_1 may possibly intersect S_n at the endpoints of each that do not intersect S_2 and S_{n-1} , respectively. A *segment* of a simple closed curve $f : [0, 1] \rightarrow \mathbb{R}^2$ is either the image $f([a, b])$ for some a, b such that $0 \leq a < b \leq 1$ or the image $f([0, a] \cup [b, 1])$ for some

a, b such that $0 < a < b < 1$.

If the vertices of a graph G can be represented by distinct points of a topological space X and every edge e in G can be represented by a simple arc joining its two endpoints in such a way that any two edges have at most an endpoint in common, then we say that G can be *embedded* in X . Such a representation of G is called an *embedding*. A graph that can be embedded in \mathbb{R}^2 is called a *planar graph*, and the image of such a graph under an embedding is called a *plane graph*.

LEMMA 3.2.6. *If Ω is an open arcwise connected subset of \mathbb{R}^2 , then any two distinct points in Ω are joined by a simple polygonal path in Ω .*

Proof. Let $x, y \in \Omega$. Because Ω is arcwise connected, there exists an arc A joining x and y in Ω . The arc A , being a continuous image of a closed interval, which is a compact set in \mathbb{R} , is compact in Ω . Since Ω is open, for each $z \in \Omega$, there exists $\varepsilon > 0$ such that the open disc around z of radius ε , abbreviated D_z for our purposes, is contained in Ω . Then the set $\{D_z | z \in A\}$ is an open cover of A , and, due to the compactness of A , it admits a finite subcover. Let $\{D_{z_1}, D_{z_2}, \dots, D_{z_k}\}$ be such a finite subcover. We can now easily choose a finite number of line segments contained within the union of the finite subcover to form a simple polygonal arc in Ω to join x and y .

PROPOSITION 3.2.7. *If G is a planar graph, then G can be embedded in the plane such that all edges are simple polygonal arcs.*

Proof. Let G be a plane graph and Γ a plane graph isomorphic to G . Let p be some vertex of Γ . Let D_p be a closed disc centered at p such that D_p intersects only the edges of Γ that are incident with p and no other vertex q of Γ is in D_p ; i.e.,

$D_p \cap D_q = \emptyset$ for every pair of distinct vertices p, q . For each edge pq of Γ , let C_{pq} be an arc contained in pq such that C_{pq} has exactly one point in each of D_p and D_q . Then C_{pq} joins D_p and D_q and has only its ends in common with $D_p \cup D_q$. Next, we redraw G including all arcs C_{pq} and replace the parts of each edge that are in the discs D_p with straight line segments joining the vertex p to the endpoint of C_{pq} on the boundary of D_p . Finally, using Lemma 3.2.6, we can replace each C_{pq} with a simple polygonal arc, and now each entire edge is a simple polygonal arc.

LEMMA 3.2.8. *Let C be a simple closed curve and P a simple polygonal arc in $\text{int}(C)$ such that P joins p and q on C and has no other point in common with C . Let P_1 and P_2 be the two arcs on C from p to q . Then $\mathbb{R}^2 \setminus (C \cup P)$ has precisely three regions whose boundaries are C , $P_1 \cup P$, and $P_2 \cup P$, respectively.*

Proof. This proof is straightforward and found in [Mohar and Thomassen].

3.3. THE JORDAN-SCHÖNFLIES THEOREM AND CONSEQUENCES

Next we need the Jordan-Schönflies Theorem.

THEOREM 3.3.1. (**Jordan-Schönflies**) *If $f : C \rightarrow C'$ is a homeomorphism between two simple closed curves C and C' in \mathbb{R}^2 , then f can be extended into a homeomorphism of all of \mathbb{R}^2 .*

Proof. Let f be the desired homeomorphism from a simple closed curve C onto a simple closed curve C' , which we can assume without loss of generality to be a convex polygon. The majority of the rest of this proof details the multiple-step process of extending of f to a homeomorphism of $\overline{\text{int}(C)}$ to $\overline{\text{int}(C')}$, meaning that it covers the curves and their interiors. Afterwards, we will see how to extend f to the exteriors of the curves, thus covering all of \mathbb{R}^2 .

Let B denote the set of points in $\text{int}(C)$ with rational coordinates, which is a countable dense set in $\text{int}(C)$. Since the points on C that are accessible from $\text{int}(C)$ are

dense in C , there exists a countable dense set in C consisting of points accessible from $\text{int}(C)$; let A be such a set. Then $A \cup B$ is also countable. Choose a sequence p_1, p_2, \dots , consisting of points in $A \cup B$ such that each point in $A \cup B$ appears in the sequence infinitely many times.

Next we construct two sequences of graphs. Let Γ_0 be a 2-connected graph consisting of C and some simple polygonal arcs in $\overline{\text{int}(C')}$, and Γ'_0 be a graph consisting of C' and some simple polygonal arcs in $\overline{\text{int}(C)}$ such that Γ_0 and Γ'_0 are plane isomorphic, with isomorphism g_0 , such that g_0 and f coincide on $C \cap V_{\Gamma_0}$. Next, we extend f to $C \cup V_{\Gamma_0}$ so that g_0 and f coincide on all of V_{Γ_0} .

To continue the sequences, we will construct 2-connected graphs satisfying the following for each $n \geq 1$:

- (a) Γ_n and Γ'_n are extensions of subdivisions of Γ_{n-1} and Γ'_{n-1} , respectively,
- (b) there is a plane isomorphism g_n of Γ_n onto Γ'_n coinciding with g_{n-1} on $V_{\Gamma_{n-1}}$,
- (c) Γ_n consists of C and simple polygonal arcs in $\overline{\text{int}(C)}$ and Γ'_n consists of C' and simple polygonal arcs in $\overline{\text{int}(C')}$
- (d) $\Gamma' - n \setminus C'$ is connected for each n .

Having done so, we can now extend f to $C \cup V_{\Gamma_n}$ such that f and g_n coincide on V_{Γ_n} . For each n , having already defined Γ_i , Γ'_i , and g_i for all $0 \leq i < n$, we define Γ_n , Γ'_n , and g_n as follows. The construction depends on the location of the point p_n from our earlier described sequence of points in $A \cup B$. As A and B are disjoint sets, p_n is in exactly one of A or B , and the procedure is broken into cases accordingly.

Supposing that $p_n \in A$, we select a simple polygonal arc P with endpoints p_n and q_n , where $q_n \in \Gamma_{n-1} \setminus C$ and P intersects Γ_{n-1} only at its endpoints. Then Γ_n will be the graph $\Gamma_{n-1} \cup P$. Note that P is contained in a face of Γ_{n-1} ; let S be the cycle bounding that face. As g_{n-1} is already defined, we refer to the face of Γ'_{n-1} that is bounded by $g_{n-1}(S)$. We will add to Γ'_{n-1} a simple polygonal arc P' as follows. If q_n is a vertex of Γ_{n-1} , then P' must join $f(p_n)$ to $g_{n-1}(q_n)$. If q_n is not a vertex of Γ_{n-1} ,

then it falls on some edge a of Γ_{n-1} , so we draw P' joining $f(p_n)$ and some point on $g_{n-1}(a)$. Then $\Gamma'_n = \Gamma'_{n-1} \cup P'$, and creating the new plane-isomorphism g_n should be clear given g_{n-1} and the graphs involved. We extend f so that $f(q_n) = g_n(q_n)$.

The next case is if $p_n \in B$. For this case, we consider the largest square that has vertical and horizontal sides, is centered at p_n , and is contained in $\overline{\text{int}(C)}$. Inside this square, we draw another square also centered at p_n and with vertical and horizontal sides, each of which has distance $< \frac{1}{n}$ from the sides of the first square. Inside the second square, we will draw a number of vertical and horizontal lines so that p_n falls on the intersection of a vertical line and a horizontal line and each of the regions into which the square is divided by these lines has diameter $< \frac{1}{n}$. We will now take the union of Γ_{n-1} and all of these vertical and horizontal line segments, possibly with the addition of a simple polygonal arc so that it is 2-connected and the portion of the union that does not intersect C is connected, and call it H_n . By Proposition 3.2.7, we can construct H_n from Γ_{n-1} by adding a sequence of paths in faces.

Adding the corresponding paths to Γ'_{n-1} gives us a graph that is plane-isomorphic to H_n which we may call H'_n . We now add vertical and horizontal lines inside $\overline{\text{int}(C')}$ to H'_n so that in the resulting graph, there are no bounded regions of diameter $\geq \frac{1}{2n}$. We want these lines to fall so that they intersect C' only on points in $f(A)$, the image of the countable dense set chosen earlier, and that each line has finite intersection with H'_n . The union of $H' - n$ with these new lines will be called Γ'_n . Note that the construction of Γ'_n from H'_n so that the former is made by adding a path, or a straight line segment within a face, requires another invocation of Proposition 3.2.7.

Returning to H_n , we add simple polygonal arcs to obtain a graph that is plane-isomorphic (by plane-isomorphism g_n) to Γ'_n , which we will be called Γ_n , continuing those sequences. We may now extend f to be defined on $C \cup V_{\Gamma_n}$ and coincides with g_n on V_{Γ_n} . Continuing in this fashion, we extend f to a one-to-one map from $C \cup V_{\Gamma_0} \cup V_{\Gamma_1} \cup \dots$ to $C' \cup V_{\Gamma'_0} \cup V_{\Gamma'_1} \cup \dots$; i.e., f is currently defined only on the

curve and vertex sets of Γ_i . However, since the domain of f at this point is dense in $\overline{\text{int}(C)}$, and its image is dense in $\overline{\text{int}(C')}$, we can now take steps to extend f to all of $\overline{\text{int}(C)}$.

Let p be a point in $\text{int}(C)$ on which f is not yet defined. Then there exists a sequence q_1, q_2, \dots of points in $V_{\Gamma_0} \cup V_{\Gamma_1} \cup \dots$ that converges to p . We claim that $f(q_1), f(q_2), \dots$ is also a convergent sequence. Let d be the distance from p to C . Let p_n be a point in B that falls within distance $\frac{d}{3}$ of p . By the earlier construction, p is inside the largest square in $\overline{\text{int}(C)}$ centered at p_n . Then Γ_n has a cycle S such that $p \in \text{int}(S)$ and S and $g_n(S)$ are both contained in discs of radius less than $\frac{1}{n}$. Since f maps $F \cap \text{int}(S)$ into $\text{int}(g_n(S))$ and $F \cap \overline{\text{ext}(S)}$ into $\overline{\text{ext}(g_n(S))}$, the entirety of the sequence $f(q_m), f(q_{m+1}), \dots$ is contained in $\text{int}(g_n(S))$ for some natural m . It follows that $f(q_1), f(q_2), \dots$ is a Cauchy sequence and is convergent, and so f is well-defined. Since f maps $\text{int}(S)$ into $\text{int}(g_n(S))$, f is continuous in $\text{int}(C)$. Similarly, since $V_{\Gamma'_0} \cup V_{\Gamma'_1}$ is dense in $\text{int}(C')$, f maps $\text{int}(C)$ onto $\text{int}(C')$, f is injective, and f^{-1} is continuous on $\text{int}(C')$.

Now that f is extended to a homeomorphism on $\text{int}(C)$, all that remains is to show that f is continuous on C to complete the extension of f onto $\overline{\text{int}(C)}$. Once f is shown to be continuous on all of $\overline{\text{int}(C)}$, the compactness of $\overline{\text{int}(C)}$ implies that f^{-1} is also continuous. We will accomplish this by considering a sequence q_1, q_2, \dots in $\text{int}(C)$ that converges to our q on C and showing that $f(q_1), f(q_2), \dots$ converges to $f(q)$ on C' .

Suppose that the image of the sequence does not converge to $f(q)$. Because $\overline{\text{int}(C')}$ is compact, the sequence of $f(q_n)$ must converge to something, which we will call $q' \neq f(q)$. Since f^{-1} is continuous on $\text{int}(C')$, $q' \in C'$. There are two arcs (of which one could be called major and one minor) on C' between q' and $f(q)$. Since A is dense in A and, therefore, $f(A)$ is dense in C' , both of these arcs contain a point of $f(A)$; call these two points $f(q_1)$ and $f(q_2)$, respectively. For some n , there is a

path P in Γ_n from q_1 to q_2 having only q_1 and q_2 in common with C . As a result of Lemma 3.2.8, P divides $\text{int}(C)$ into two regions, as $g_n(P)$ also divides $\text{int}(C')$ into two regions. One of the two regions of $\text{int}(C')$ contains almost all of the $f(q_n)$, and the other has $f(q)$ on the boundary that it does not share with the other region. Therefore, $\lim_{n \rightarrow \infty} f(q_n)$ can not be q' , and therefore it must be $f(q)$. It follows that f extends as a homeomorphism to $\text{int}(C)$.

Next, we extend f to $\text{ext}(C)$, which extends f to the entire plane \mathbb{R}^2 . At this point, we superimpose the two curves into a standard coordinate plane and assume without loss of generality that C contains the origin in its interior and both C and C' are in the interior of a quadrilateral T with corners at $(-1, 1)$, $(1, 1)$, $(1, -1)$, and $(-1, -1)$. Let L_1 , L_2 , and L_3 be line segments with endpoints at $(1, 1)$, $(-1, -1)$, and $(1, -1)$, respectively, and opposite endpoints on C , such that each segment is a part of a line through the origin. Refer to the endpoints of L_i on C as p_i . Let L'_1 be a polygonal arc from $f(p_1)$ to $(1, 1)$ and L'_2 be a polygonal arc from $f(p_2)$ to $(-1, -1)$, chosen so that L'_1 and L'_2 do not intersect and neither has any points other than endpoints in common with either T or C' . Next, we can find a polygonal arc L'_3 from $f(p_3)$ to either $(1, -1)$ or $(-1, 1)$ such that L'_3 does not intersect L'_1 or L'_2 and has only endpoints in common with C' and T . If L'_3 has its endpoint at $(-1, 1)$, then we reflect C' over the line $y = x$ so that L'_3 goes to $(1, -1)$ instead. Following the same technique as the first part of the proof, we extend f to a homeomorphism of $\overline{\text{int}(T)}$ such that f is the identity on T . Then f can be naturally extended to a homeomorphism of \mathbb{R}^2 such that f is the identity on the remaining $\text{ext}(T)$.

THEOREM 3.3.2. *If $G = (V, E, st)$ is a 2-connected plane graph, then G has $|E| - |V| + 2$ faces, each of which is bounded by a cycle of G . If G' is a plane graph isomorphic to G such that each facial cycle in G corresponds to a facial cycle in G' and the cycle bounding the outer face in G corresponds to the boundary of the outer*

face in G' , then any homeomorphism of G onto G' (extending the isomorphism of G onto G') can be extended into a homeomorphism of the entire plane.

Proof. This proof requires an assemblage of minor lemmas throughout the text of [Thomassen and Mohar] and is omitted.

PROPOSITION 3.3.3. *Let P_1, P_2, P_3 be simple arcs with ends p, q such that $P_i \cap P_j = \{p, q\}$ for $1 \leq i < j \leq 3$. Then $P_1 \cup P_2 \cup P_3$ has precisely three faces with boundaries $P_1 \cup P_2, P_1 \cup P_3$, and $P_2 \cup P_3$, respectively. If the outer face is bounded by $P_1 \cup P_2$, then P'_1, P'_2, P'_3 are simple arcs joining p', q' such that $P'_i \cap P'_j = \{p', q'\}$ for $1 \leq i < j \leq 3$, and such that $P'_3 \subset \overline{\text{int}}(P'_1 \cup P'_2)$, then any homeomorphism f of $P_1 \cup P_2 \cup P_3$ onto $P'_1 \cup P'_2 \cup P'_3$ can be extended to a homeomorphism of \mathbb{R}^2 onto itself.*

Proof. If $P_1 \subset \overline{\text{ext}}(P_2 \cup P_3), P_2 \subset \overline{\text{ext}}(P_1 \cup P_3)$, and $P_3 \subset \overline{\text{ext}}(P_1 \cup P_2)$, then it is easy to extend $P_1 \cup P_2 \cup P_3$ to a $K_{3,3}$ in the plane, which is a contradiction. We may assume without loss of generality that $P_3 \subset \overline{\text{int}}(P_1 \cup P_2)$. The first part of the proposition follows easily. To prove the last part, it is sufficient to consider the case where P'_1, P'_2, P'_3 are polygonal arcs. This case is done using the Jordan-Schönflies Theorem to $\overline{\text{int}}(P_1 \cup P_3), \overline{\text{int}}(P_2 \cup P_3)$, and $\overline{\text{ext}}(P_1 \cup P_2)$, respectively.

3.4. EVERY COMPACT SURFACE ADMITS A TRIANGULATION

The last result that we need before stating and proving the main result of this chapter, that any compact surface can be triangulated, is the following. We will introduce some terminology in this lemma to facilitate understanding.

LEMMA 3.4.1. *Let $\gamma : [0, 1] \rightarrow \mathbb{R}^2, \gamma_2 : [0, 1] \rightarrow \mathbb{R}^2$, and $\gamma_3 : [0, 1] \rightarrow \mathbb{R}^2$ be closed simple continuous curves and assume that $\gamma_3([0, 1]) \subset \overset{\circ}{\gamma_2([0, 1])}$ (the curve described by γ_3 is in the interior of the curve described by γ_2 .) Define a bad segment of γ as a*

segment, P , joining two points p, q on $\gamma_2[(0, 1)]$ with all other points on $\gamma_2(\overset{\circ}{[0, 1]})$ and define a very bad segment as a bad segment that intersects $\gamma_3([0, 1])$. Then there are only finitely many very bad segments.

Proof. Because γ is continuous and $[0, 1]$ is compact, $\gamma([0, 1])$ is also compact. We have already verified that there exists $\varepsilon > 0$ such that $\gamma_3([0, 1])$ can be covered by a finite number of open discs with radius ε centered on $\gamma_3[0, 1]$, all of which are contained inside $\gamma_2(\overset{\circ}{[0, 1]})$. Suppose that there are infinitely many very bad segments. Then we can choose an infinite sequence of very bad segments, $P_1, P_2, \dots, P_n, \dots$. For each n , consider the endpoints of P_n , which we will call p_n and q_n . There exist $u_n, v_n \in [0, 1]$ such that $\gamma(u_n) = p_n$ and $\gamma(v_n) = q_n$. Use these values to construct another infinite sequence $\{t_k\}$, where for all $k \geq 1$, $t_{2k-1} = u_k$ and $t_{2k} = v_k$. (The odd-indexed values of t_k are from u_k , and the even-indexed values are from v_k - $t_1 = u_1, t_2 = v_1, t_3 = u_2$, etc.) As all values of t_k are contained within the compact set $[0, 1]$, the sequence must have some accumulation point t . Since γ is continuous, the sequences $\{p_n\}$ and $\{q_n\}$ each have subsequences converging to $\gamma(t)$. Since $q_n \in \gamma_2([0, 1])$ for all n and $\{q_n\}$ has a subsequence that converges to $\gamma(t)$, it follows that $\gamma(t) \in \gamma_2([0, 1])$; i.e., $\gamma(t)$ is a point of intersection between γ and γ_2 . Because γ is continuous, for every $\eta > 0$, there exists $\varepsilon_2 > 0$ such that $\gamma(u) \in B(\gamma(t), \eta)$ for all u such that $|u - t| < \varepsilon_2$. It follows that for infinitely many n , $P_n \subset B(\gamma(t), \eta)$. Choosing η to be less than ε will keep P_n from intersecting γ_3 for some n , which contradicts the fact that P_n is a very bad segment for all n . Therefore, there must be only finitely many very bad segments.

DEFINITION 3.4.2. Let \mathcal{P} be a finite set of pairwise disjoint convex polygons with their interiors in the plane such that all sides have the same length. Let S be a topological space obtained by gluing polygons in \mathcal{P} in such a way that every edge of a polygon $P \in \mathcal{P}$ is identified with exactly one other edge of either P or some other

polygon in \mathcal{P} . Using each corner of a polygon as a vertex and the sides of polygons as edges, with incidence and adjacency defined in the obvious way, we construct a graph G . If S is a connected surface, then we say that G is a 2-cell embedding of S . If all $P \in \mathcal{P}$ are triangles, then G is called a triangulation of S and S is called a triangulated surface.

THEOREM 3.4.3. *Every connected compact surface S is homeomorphic to a triangulated surface.*

Proof. Let S be a compact connected surface. It suffices to show that S is homeomorphic to some surface with a 2-cell embedding. Once we know that it can be expressed as a set of polygons as described above, it is easy to triangulate the polygons. By definition, each point $p \in S$ has a neighborhood U_p open in S that is homeomorphic to an open disc $D(p)$ in \mathbb{R}^2 . Let $\theta_p : D(p) \rightarrow U_p$ be that homeomorphism. Inside each D_p , insert two quadrilaterals $Q_1(p)$ and $Q_2(p)$ such that $Q_1(p) \subseteq Q_2(p)$ and $p \in \overset{\circ}{Q}_2(p)$. Obviously, the union $\cup_{p \in S} \theta_p(Q_1(p))$ covers all of S . Since S is compact, though, there is a finite set $\{p_1, p_2, \dots, p_n\}$ such that $\cup_{i=1}^n \theta_{p_i}(\overset{\circ}{Q}_1(p_i))$ covers S . For these p_i , we will now choose new $D(p_i)$, doing so in such a way that they are pairwise disjoint, which is possible because they are open sets in the Hausdorff space \mathbb{R}^2 . Doing so gives rise to new homeomorphisms θ_{p_i} which map the $D(p_i)$ to open sets U_{p_i} and, of course, new quadrilaterals $Q_1(p_i)$ fitting inside them. The claim is that the set of $Q_1(p_i)$ can be chosen in such a way as to form a 2-cell embedding of S , completing the proof. Specifically, suppose by induction on k that $Q_1(p_1), Q_1(p_2), \dots, Q_1(p_{k-1})$ have been chosen so that any two of $\theta_{p_1}(Q_1(p_1)), \theta_{p_2}(Q_1(p_2)), \dots, \theta_{p_{k-1}}(Q_1(p_{k-1}))$ have only a finite number of intersection points in S .

As in the previous lemma, we define a *bad segment* to be a segment P of some $Q_1(p_j)$ where $1 \leq j \leq k-1$ such that $\theta_{p_j}(P)$ has its endpoints on two points of $\theta_{p_k}(\overset{\circ}{Q}_2(p_k))$ and all other points in $\theta_{p_k}(\overset{\circ}{Q}_2(p_k))$. We now choose, for each k , another

quadrilateral $Q_3(p_k)$ such that $Q_1(p_k) \subseteq Q_3(p_k)$ and $Q_3(p_k) \subseteq Q_2(p_k)$; i.e., $Q_3(p_k)$ is between $Q_1(p_k)$ and $Q_2(p_k)$. Each bad segment P exists in $Q_1(p_j)$ for some j . Consider $\theta_{p_k}^{-1}(\theta_{p_j}(P))$, the inverse image through θ_{p_k} of its image through θ_{p_j} , which is contained in $Q_2(p_k)$ by design. We will define a *very bad segment* as a bad segment P for which $\theta_{p_j}(P)$ intersects $\theta_{p_k}(Q_3(p_k))$. By the previous lemma, since $Q_3(p_k) \subseteq Q_2(p_k)$, there are only finitely many very bad segments.

Take all very bad segments $\theta_{p_k}^{-1}(\theta_{p_j}(P))$ inside $Q_2(p_k)$ along with $Q_2(p_k)$ itself to yield a 2-connected plane graph Γ . As we proved in (Proposition E.3), it is possible to create a graph Γ' inside $Q_2(p_k)$ that is homeomorphic and graph isomorphic to Γ , but all edges of Γ' are simple polygonal arcs. As we proved in (Theorem E.2 / 2.2.3), we can extend the plane isomorphism from Γ to Γ' to a homeomorphism of the closure of $Q_2(p_k)$, keeping $Q_2(p_k)$ fixed and mapping $Q_1(p_k)$ and $Q_3(p_k)$ to simple closed curves Q'_1 and Q'_3 , respectively, such that $\overset{\circ}{Q}'_1 \subseteq \overset{\circ}{Q}'_2$ and $p_k \in \theta_{p_k}(\overset{\circ}{Q}'_1)$.

At this point, we find a simple closed polygonal curve $Q''_3 \subseteq Q_2(p_k)$ whose interior contains Q'_1 such that all bad segments in $Q_2(p_k)$ that are intersected by Q''_3 are very bad segments, which we have turned into simple polygonal arcs. For each point $q \in Q'_3$, let $R(q)$ be a square centered at q such that $R(q)$ does not intersect any bad segment that is not very bad, nor does it intersect Q'_1 . By compactness of Q'_3 , we can choose a finite covering of Q'_3 , and we will choose the minimal such covering. The union of these squares is a 2-connected plane graph. The outer cycle of this graph will serve as our Q''_3 .

Now we have the graph $\Gamma' \cup Q''_3$, which is 2-connected. We can redraw this graph so that Q''_3 is replaced by an isomorphic copy that is a quadrilateral with $Q'_1 \subseteq \overset{\circ}{Q}''_3$. Using (Theorem E.2), we can extend this isomorphism to \mathbb{R}^2 . If we let Q''_3 be the new choice of $Q_1(p_k)$, then any two of $\theta_{p_1}(Q_1(p_1)), \theta_{p_2}(Q_1(p_2)), \dots, \theta_{p_k}(Q_1(p_k))$ have only finite intersection. The inductive hypothesis is proved for all k .

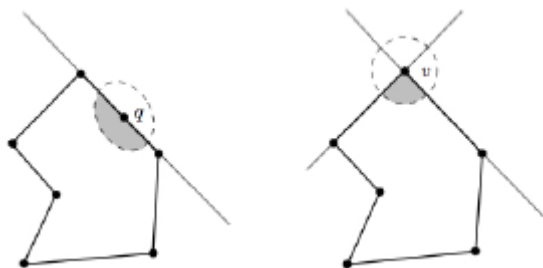
Now we can assume that there are only finitely many very bad segments inside each $Q_2(p_k)$ and that those very bad segments are simple polygonal arcs that form a 2-connected plane graph. The union $\cup_{i=1}^n \theta_{p_i}(Q_1(p_i))$ may be thought of as a graph Γ drawn on S . Each region of $S \setminus \Gamma$ is bounded by a cycle C in Γ . We may think of this cycle C as a simple closed polygonal curve inside some $Q_2(p_i)$. Now we draw a convex polygon C' of side length 1 such that the corners of C' correspond to the vertices of C . After appropriate identification of the sides of the polygons C' corresponding to the faces of Γ in S , we get a surface S' with a 2-cell embedding, Γ' , which is isomorphic to Γ . This isomorphism of Γ to Γ' may be extended to a homeomorphism f of the point set of Γ on S onto the point set of Γ' on S' . In particular, the restriction of f to the above cycle, C , is a homeomorphism of C onto C' . We have already shown that f can be extended to a homeomorphism of $\overline{\text{int}((C))}$ to $\overline{\text{int}(C')}$. This defines a homeomorphism of S onto S' , completing the proof that any surface S is homeomorphic to a triangulated surface S' .

CHAPTER 4

POLYGONAL PRESENTATIONS AND THE CLASSIFICATION THEOREM

4.1. POLYGONAL REGIONS

In the previous chapter, we explored the representation of surfaces via simplicial complexes and triangulation. The next step is to generalize that idea by representing surfaces using polygons of more than three sides. We begin by formally defining a polygon.



DEFINITION 4.1.1. *A subset P of the plane is a polygonal region if it is a compact subset whose boundary is a 1-dimensional simplicial complex satisfying the following:*

- (1) *Each point q of an edge that is not a vertex has a neighborhood $U \subset \mathbb{R}^2$ such that $P \cap U$ is equal to the intersection of U with a closed half-plane $\{(x, y) | ax + by + c \geq 0\}$.*

- (2) *Each vertex v has a neighborhood $V \subset \mathbb{R}^2$ such that $P \cap V$ is equal to the intersection of V with two closed half-planes whose boundaries intersect only at v .*

We can represent a surface with a polygonal region, provided the polygon has an even number of edges, by making identifications on pairs of edges with the quotient topology as we did with the unit interval product $I \times I$ in Chapter 2.

THEOREM 4.1.2. *Let P be a polygonal region in the plane with an even number of edges and suppose we are given an equivalence relation that identifies each edge with exactly one other edge by means of a simplicial isomorphism. Then the resulting quotient space is a compact surface.*

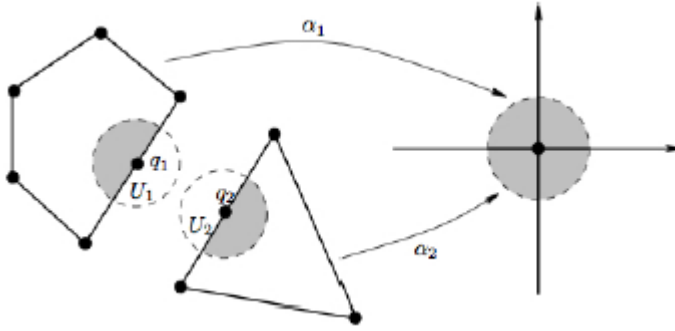
Proof. Let M be the quotient P/\sim and let $\pi : P \rightarrow M$ denote the quotient map. Since P is compact, $f(P) = M$ is also compact as the continuous image of a compact space. We must show that M is locally Euclidean and Hausdorff. The equivalence relation \sim identifies only edges with edges and vertices with vertices, so each point of M falls into one of three categories, referred to as follows:

- (1) *face points*, whose inverse images in P are in $\text{Int } P$,
- (2) *edge points*, whose inverse images are on edges but not vertices, and
- (3) *vertex points*, whose inverse images are vertices.

We will consider these three different cases individually to prove that M is locally Euclidean.

As a quotient map, π is surjective, and π is injective on $\text{Int } P$, so π is bijective on $\text{Int } P$. By the closed map lemma, π is a homeomorphism on $\text{Int } P$. Then $\pi(\text{Int } P) \cong \text{Int } P \cong \mathbb{R}^2$, so $\pi(\text{Int } P)$ is a Euclidean neighborhood of every face point. For any edge point q , choose a sufficiently small neighborhood that contains no vertex points. Each point q has exactly two inverse images, q_1 and q_2 , each on a different

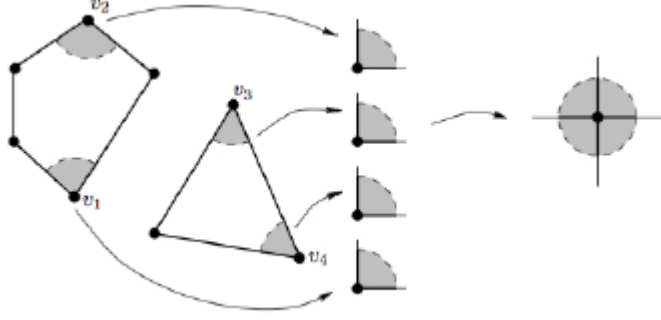
edge. There exist disjoint neighborhoods U_1 of q_1 and U_2 for q_2 such that $P \cap U_1$ and $P \cap U_2$ are disjoint half-disks. Let $V_1 = P \cap U_1$ and $V_2 = P \cap U_2$. We can construct affine homeomorphisms α_1 and α_2 such that α_1 maps V_1 to a half-disk on the upper half plane, α_2 maps V_2 to a half disk on the lower half plane, q_1 and q_2 are both mapped to the origin, and boundary identifications are respected. If necessary, we can shrink V_1 and V_2 until $V_1 \cup V_2$ is a saturated open set in the plane; i.e., for every point in $V_1 \cap \partial P$, the corresponding boundary point is in V_2 and vice versa. We may now define another homeomorphism $\alpha : V_1 \cup V_2 \rightarrow \mathbb{R}^2$ such that $\alpha(q) = \alpha_1(q)$ for all $q \in V_1$ and $\alpha(q) = \alpha_2(q)$ for all $q \in V_2$. Define an equivalence relation \sim so that for any edge points r_1 and r_2 in V_1 and V_2 , respectively, $r_1 \sim r_2$ when $\alpha(r_1) = \alpha(r_2)$. Since $\pi|_{V_1 \cup V_2}$ is a quotient map, and α is a quotient map onto an open ball centered at the origin and making the same identifications as π . By the uniqueness of the quotient map, the quotient spaces are homeomorphic, so every edge point has a Euclidean neighborhood.



The process for

showing that a vertex point has a Euclidean neighborhood is similar. Rather than having exactly two inverse images, each vertex point v has a finite set of vertices $\{v_1, v_2, \dots, v_k\}$ that map to it. Consider an open ball centered at the origin and a sector of that ball with angle measure $\frac{2\pi}{k}$. For each i , we can choose a homeomorphism from a neighborhood of v_i in P to the sector. We create a ball containing a neighborhood of the origin by rotating and piecing together k copies of the sector,

taking care to see that all sectors are appropriately scaled and that edge identifications are respected. The same result follows, so M is locally Euclidean.



Since M is the quotient space of the quotient map from the polygonal region P , the preimage of any pair of distinct points in M can be separated into disjoint open sets by choosing sufficiently small open balls, the image of which will be open sets in M separating the two points in M . Since M is locally Euclidean and Hausdorff, M is a surface.

It turns out that the converse of this theorem, the fact that every compact surface is the quotient space of a polygon with sides pairwise identified, is also true. Before stating and proving that result, we must develop the notion of a polygonal presentation, which will facilitate describing these surfaces in a uniform manner.

4.2. POLYGONAL PRESENTATION

DEFINITION 4.2.1. *Let S be a finite set of letters or symbols. A polygonal presentation $\mathcal{P} = \langle S | W_1, W_2, \dots, W_k \rangle$ is a finite list of words W_i , each of which is an ordered k -tuple ($k \geq 3$) of symbols of the form a or a^{-1} , where $a \in S$.*

When describing a presentation, we leave out the curly braces surrounding the elements of S and juxtapose elements of S to represent words, which are separated

by commas. For example, if $S = \{a, b\}$, and we had two words, $W_1 = \{aba^{-1}b^{-1}\}$ and $W_2 = \{aa\}$, then we could write $\mathcal{P} = \langle a, b | aba^{-1}b^{-1}, aa \rangle$.

DEFINITION 4.2.2. *Given a polygonal presentation \mathcal{P} , the geometric realization of \mathcal{P} , denoted $|\mathcal{P}|$, is a topological space constructed by the following algorithm:*

- (1) *For each word W_i , let P_i be a regular convex k -sided polygon centered at the origin with side length 1 and one vertex on the positive y -axis.*
- (2) *Define a bijective function between the symbols of W_i and the edges of P_i in a counterclockwise order, beginning at the aforementioned vertex on the y -axis.*
- (3) *Let $|\mathcal{P}|$ denote the quotient space of $\coprod_i P_i$ determined by identifying edges that have the same symbol by an affine homeomorphism that matches the initial vertices and terminal vertices of any pairs of edges if both are labeled a or both are labeled a^{-1} and initial vertices to terminal vertices for any pair of edges labeled a and a^{-1} .*

In the special case where W_i is a word of length 2, we define P_i to be a sphere if the word is aa^{-1} or $a^{-1}a$ and a projective plane if the word is aa or $a^{-1}a^{-1}$.

If two presentations \mathcal{P}_1 and \mathcal{P}_2 have homeomorphic geometric realizations, then we can say that they are *topologically equivalent* and write $\mathcal{P}_1 \cong \mathcal{P}_2$.

Note that these geometric realizations of polygonal presentations are topological spaces, but if we want a geometric realization to be a compact surface, it must satisfy the conditions set forth by Theorem 4.1.2. If it does meet these requirements, then we will refer to it as a *surface presentation*.

DEFINITION 4.2.3. *A surface presentation is a polygonal presentation*

$$\mathcal{P} = \langle S | W_1, W_2, \dots, W_k \rangle$$

such that for each symbol $a \in S$, there are exactly two occurrences of either a or a^{-1} in W_1, W_2, \dots, W_k .

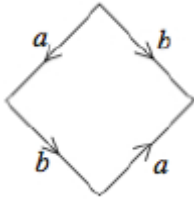
To review, the basic surfaces we have mentioned are the following, with their polygonal presentations:

$$\mathbb{S}^2 = \langle a | aa^{-1} \rangle \text{ or } \langle a, b | abb^{-1}a^{-1} \rangle$$

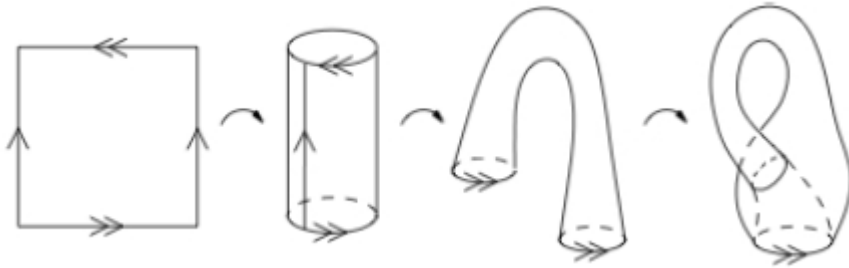
$$\mathbb{T}^2 = \langle a, b | aba^{-1}b^{-1} \rangle$$

$$\mathbb{P}^2 = \langle a | aa \rangle \text{ or } \langle a, b | abab \rangle$$

In addition to these main three surfaces, we have \mathbb{K} , the Klein bottle. We will define \mathbb{K} to be the manifold that comes from applying the equivalence relation generated by $(0, t) \sim (1, t)$ and $(t, 0) \sim (1 - t, 0)$ for $0 \leq t \leq 1$ to $I \times I$.



We can visualize this construction by making a cylinder out of $I \times I$ and then twisting it before reattaching in the same manner as making the torus. Its polygonal presentation, then, is given as $\langle a, b | abab^{-1} \rangle$.



With this notation in hand, we are now ready to prove the converse of Theorem 4.1.2.

THEOREM 4.2.4. *Every compact surface admits a polygonal presentation.*

Proof. Let M be a compact surface. By the triangulation theorem from the previous chapter, M is homeomorphic to a 2-dimensional simplicial complex K in which each 1-simplex is a face of exactly two different 2-simplices. From K , construct a surface presentation \mathcal{P} such that each 2-simplex generates a word of length 3 and edges are labeled with the same symbol if they come from the same 1-simplex. Let $P = P_1 \amalg P_2 \amalg \dots \amalg P_k$. Then we have two quotient maps: $\pi_K : P \rightarrow |K|$ and $\pi_{\mathcal{P}} : P \rightarrow |\mathcal{P}|$. It suffices to show that both maps make the same identifications.

It is clear by construction that the two quotient maps identify the same edges, so it remains to show that they identify vertices with instructions from the edge identifications. Any vertex $v \in K$ must be in some 1-simplex, or edge. By the triangulation theorem, this edge must be in two 2-simplices σ and σ' . Now we define an equivalence relation on the set of 2-simplices containing v , σ and σ' , are equivalent if there exists a sequence of 2-simplices $\sigma = \sigma_1, \sigma_1 = \sigma_2, \dots, \sigma_k = \sigma'$ such that σ_i shares an edge with σ_{i+1} for $i = 1, 2, \dots, k-1$. To prove that π_K and $\pi_{\mathcal{P}}$ identify the same vertices, it is sufficient to show that there is only one equivalence class.

Suppose, for the sake of a contradiction, that there are two different equivalence classes $\{\sigma_1, \sigma_2, \dots, \sigma_k\}$ and $\{\tau_1, \tau_2, \dots, \tau_m\}$ such that $\sigma_i \sim \sigma_j$ and $\sigma_i \not\sim \tau_j$ for all i and j . Let ε be small enough that $B_\varepsilon(v)$ intersects only simplices that contain v . Then, since $B_\varepsilon(v) \cap |K|$ is an open subset of $|K|$, v has a neighborhood U that is homeomorphic to \mathbb{R}^2 that is also a subset of $B_\varepsilon v \cap |K|$. Since this neighborhood is homeomorphic to \mathbb{R}^2 , $U \setminus \{v\}$ is connected. If we assume that there are two distinct equivalence classes, then $U \cap (\sigma_1 \cup \sigma_2 \cup \dots \cup \sigma_k) \setminus \{v\}$ and $U \cap (\tau_1 \cup \tau_2 \cup \dots \cup \tau_m) \setminus |K|$ are disjoint open sets in $|K|$. Then $U \setminus \{v\} = (U \cap (\sigma_1 \cup \sigma_2 \cup \dots \cup \sigma_k) \setminus \{v\}) \cup (U \cap (\tau_1 \cup \tau_2 \cup \dots \cup \tau_m) \setminus \{v\})$ is disconnected, which yields the desired contradiction. There is only one equivalence

class, so the two quotient maps make the same identifications, and $|K|$ is homeomorphic to $|\mathcal{P}|$.

The following lemma allows us to see when two different polygonal presentations yield homeomorphic geometric realizations.

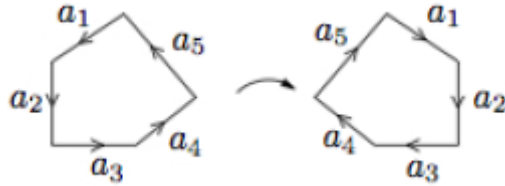
LEMMA 4.2.5. *Let P_1 and P_2 be convex polygons with the same number of edges, and let $f : \partial P_1 \rightarrow \partial P_2$ be a simplicial isomorphism. Then f extends to a homeomorphism $F : P_1 \rightarrow P_2$.*

Proof. Choose any points $p_1 \in \text{Int } P_1$ and $p_2 \in \text{Int } P_2$. Convexity of the polygons tells us that the line segment from p_i to each vertex of P_i is contained entirely in P_i . For each pair of adjacent vertices of P_i , the convex hull spanned by p_i and the vertices is a 2-simplex. The disjoint union of these simplices with interior line segments and their endpoints identified forms a simplicial complex whose polyhedron is P_i . Then $F : P_1 \rightarrow P_2$ is the simplicial map whose restriction to ∂P_1 is f and that takes p_1 to p_2 .

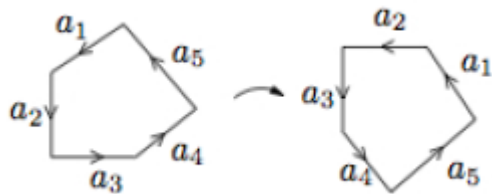
Now we have a list of *elementary transformations* that can be performed on a polygonal presentation and still yield the same surface. For the following, we adopt the conventions that e denotes a symbol not in S , W_1W_2 denotes a word formed by concatenating words W_1 and W_2 , and $(a^{-1})^{-1} = a$.

- (1) Reflecting: If $a_1a_2 \dots a_n$ is a word in the presentation, it may be rewritten as

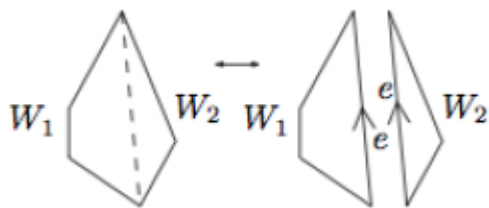
$$an^{-1} \dots a_2^{-1}a_1^{-1}.$$



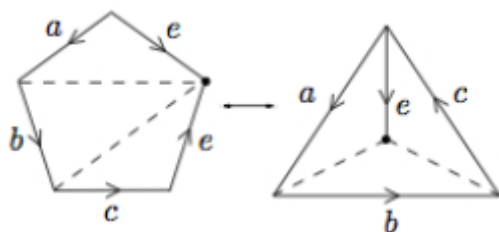
- (2) Rotating: If $a_1a_2\dots a_n$ is a word in the presentation, it may be rewritten as $a_2\dots a_na_1$.



- (3) Cutting: If W_1 and W_2 both have length at least 2, replace W_1W_2 with the two words $W_1e, e^{-1}W_2$.
- (4) Pasting: If W_1e and $e^{-1}W_2$ are words in the presentation, they may be combined along e to give W_1W_2 as a word in the presentation.



- (5) Folding: If W_1ee^{-1} is a word in the presentation, it may be replaced with W_1 .
- (6) Unfolding: If W_1 is a word in the presentation and e is a symbol that does not already appear, W_1 may be rewritten as W_1ee^{-1} .



THEOREM 4.2.6. *Any elementary transformation of a polygonal presentation produces a geometric realization topologically equivalent to that of the original polygonal presentation.*

Proof. Note that cutting and pasting are symmetric, so we only need to prove that one of them presents a geometric realization homeomorphic to the original. The same is true of folding and unfolding.

- (1) Reflecting: Let P_1 be the geometric realization of a_1, a_2, \dots, a_n and P'_1 be the geometric realization of $a_n^{-1}, \dots, a_2^{-1}, a_1^{-1}$. Reflection is a linear transformation, so we can choose the reflection matrix as our homeomorphism. It is clear that it is bijective and continuous. We can extend the homeomorphism to W_2, \dots, W_K by the identity map.
- (2) Rotating: Let P_1 be the geometric realization of a_1, a_2, \dots, a_n and P'_1 be the geometric realization of a_2, \dots, a_n, a_1 . Similar to reflecting, we can choose a rotation matrix as our homeomorphism. This linear transformation is also clearly bijective and continuous, and we can extend the homeomorphism by the identity map once again.
- (3) Cutting: Let P_1 and P_2 be polygons labeled W_1e and $e^{-1}W_2$, respectively, and let P' be the polygon labeled W_1W_2 . Let $\pi : P_1 \amalg P_2 \rightarrow S$ and $\pi' : P' \rightarrow S$ be the two necessary quotient maps. Since P' is convex, the line segment from the terminal vertex of W_1 to the initial vertex is contained in P' ; call it e . The continuous map $f : P_1 \amalg P_2 \rightarrow P'$ takes each edge of P_i to its corresponding edge in P' and identifies e and e^{-1} . By the closed map lemma, f is a quotient map. Since $\pi' \circ f$ and π make the same identifications from the same domain, S and S' are homeomorphic by uniqueness of the quotient map. If the polygonal presentation has any other words, then the homeomorphism can again be extended by the identity map.

(4) Folding: We can assume without loss of generality that $W_1 = abc$ has length at least 3. Let P and P' be polygons $abcee^{-1}$ and abc , respectively. Let $\pi : P \rightarrow S$ and $\pi' : P' \rightarrow S'$ be the two quotient maps. By adding edges along the diagonals, we can turn P into a simplicial complex represented by the words $e^{-1}ad, d^{-1}bf$, and $f^{-1}ce$, where sides of the same letter are identified and the vertex identification is forced by the edge identification. Let $f : P \rightarrow P'$ be the simplicial map that takes edges in P to edges with the same label in P' . Then $\pi \circ f$ and π are quotient maps that make the same identifications, so S and S' are homeomorphic by uniqueness of the quotient map. Again, we can extend the homeomorphism to any other words in the presentation by the identity.

We can express the polygonal presentation of a connected sum of two surfaces using concatenation.

THEOREM 4.2.7. *Let M_1 and M_2 be surfaces that admit polygonal presentations $\langle S_1|W_1 \rangle$ and $\langle S_2|W_2 \rangle$, respectively, in which S_1 and S_2 are disjoint sets and each presentation has a single face. Then the connected sum $M_1 \# M_2$ admits the presentation $\langle S_1, S_2|W_1 W_2 \rangle$.*

Proof. Beginning with the presentation of M_1 , we can employ the cutting procedure three times to show that $\langle S_1|W_1 \rangle \cong \langle S_1, a, b, c|W_1 c^{-1} b^{-1} a^{-1}, abc \rangle$. The word abc represents a polygon, the convex hull of which is a 2-simplex that is homeomorphic to a closed ball. Let B_1 be the interior of the convex hull of the polygon corresponding to the word abc . Then the geometric realization of $\langle S_1, a, b, c|W_1 c^{-1} b^{-1} a^{-1} \rangle$ is homeomorphic to $M_1 \setminus B_1$, which was previously defined to be M'_1 . Similarly, M'_2 has presentation $\langle S_2, a, b, c|abc W_2 \rangle$. Then the presentation $\langle S_1, S_2, a, b, c|W_1 c^{-1} b^{-1} a^{-1}, abc W_2 \rangle$ is $M'_1 \amalg M'_2$ with the boundaries of the removed ball represented by abc identified. By definition, this space is $M_1 \# M_2$. Pasting along c and folding a and b gives us

$$\langle S_1, S_2 | W_1 W_2 \rangle.$$

We can therefore add the following to our list of surface presentations:

- (1) The connected sum of n tori: $\langle a_1, b_1, \dots, a_n, b_n | a_1 b_1 a_1^{-1} b_1^{-1} \dots a_n b_n a_n^{-1} b_n^{-1} \rangle$
- (2) The connected sum of n projective planes: $\langle a_1, \dots, a_n | a_1 a_1 \dots a_n a_n \rangle$

4.3. THE CLASSIFICATION THEOREM

Now we can state our main result.

THEOREM 4.3.1. *Any surface presentation is equivalent by a sequence of elementary transformations to a presentation of one of the following:*

- \mathbb{S}^2
- $\mathbb{T}^2 \# \dots \# \mathbb{T}^2$
- $\mathbb{P}^2 \# \dots \# \mathbb{P}^2$

And so every compact surface is homeomorphic to one of these surfaces.

Before presenting the proof, we recognize that a few well known surfaces are missing from the list. Consider \mathbb{K} , the Klein bottle:

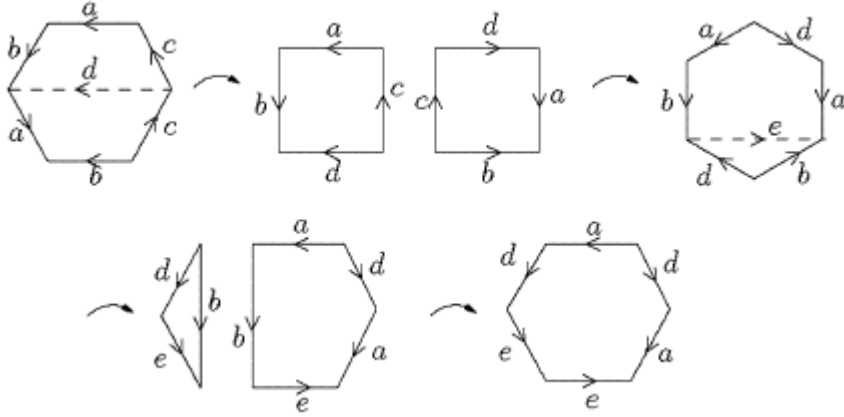
$$\begin{aligned} \langle a, b | abab^{-1} \rangle &\approx \langle a, b, c | abc, c^{-1}ab^{-1} \rangle \\ &\approx \langle a, b, c | bca, a^{-1}cb \rangle \\ &\approx \langle b, c | bbcc \rangle, \end{aligned}$$

which we have seen is the connected sum $\mathbb{P}^2 \# \mathbb{P}^2$.



If we have a connected sum involving both tori and projective planes, such as $\mathbb{P}^2 \# \mathbb{T}^2$, a surprising result follows. Consider three \mathbb{P}^2 s in a connected sum. This is the same as $\mathbb{K} \# \mathbb{P}^2$, so it is:

$$\begin{aligned}
& \langle a, b, c | abab^{-1}cc \rangle \\
& \approx \langle a, b, c, d | bab^{-1}cd, d^{-1}ca \rangle \\
& \approx \langle a, b, d | dbab^{-1}da^{-1} \rangle \\
& \approx \langle a, b, d, e | dbe, e^{-1}ab^{-1}da^{-1} \rangle \\
& \approx \langle a, b, d, e | edb, b^{-1}da^{-1}e^{-1}a \rangle \\
& \approx \langle a, d, e | edda^{-1}e^{-1}a \rangle \\
& \approx \langle a, d, e | dda^{-1}e^{-1}ae \rangle \\
& = \mathbb{P}^2 \# \mathbb{T}^2.
\end{aligned}$$



Having established that these obvious surfaces are subject to the classification at hand, we will now proceed to prove the theorem.

Proof of the Classification Theorem. We will begin with a polygonal presentation of

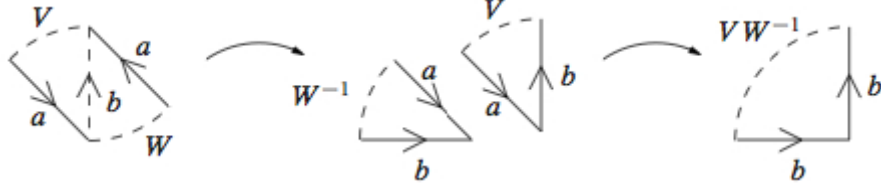
an arbitrary connected compact surface M and show that by a sequence of elementary transformations, the surface is homeomorphic to one of the items listed in the statement of the theorem. A suitable algorithm follows.

- (1) The surface M admits a presentation that has only one face. Since M is connected, if there are two or more faces, then each face must have an edge that is identified with an edge in a different face. Repeated pasting transformations, along with rotations and reflections as necessary, will yield a presentation with only one face.
- (2) If M is not a sphere, then M admits a presentation with no adjacent complementary pairs. Any adjacent complementary pairs can be removed by folding. The only instance in which an adjacent complementary pair could not be removed by folding is that in which the pair is the only pair of edges in the presentation. This presentation is of the form $\langle a|aa^{-1} \rangle$, which is a sphere. In this case, we have shown that M is classified as homeomorphic to \mathbb{S}^2 . We may assume now that M is not homeomorphic to a sphere.
- (3) There is a presentation of M in which all twisted pairs are adjacent. Suppose we have a twisted pair that is not adjacent. Then the presentation takes the form $VaWa$, where V and W are both non-empty strings of edges. The following sequence accomplishes this goal:

$$\begin{aligned}
\langle a, V, W|VaWa \rangle &\cong \langle a, b, V, W|Vab, b^{-1}Wa \rangle \\
&\cong \langle a, b, V, W|bVa, a^{-1}W^{-1}b \rangle \\
&\cong \langle b, V, W|VW^{-1}bb \rangle.
\end{aligned}$$

In this last presentation, no adjacent twisted pairs have been separated, and one non-adjacent twisted pair has been replaced by an adjacent pair. It is

possible that we may have introduced a new non-adjacent twisted pair, but finitely many repetitions of this step will eventually eliminate all non-adjacent twisted pairs. At this point, we may have reintroduced some adjacent complementary pairs, in which case we should repeat step 2. Note that step 2 will not introduce any new non-adjacent twisted pairs.



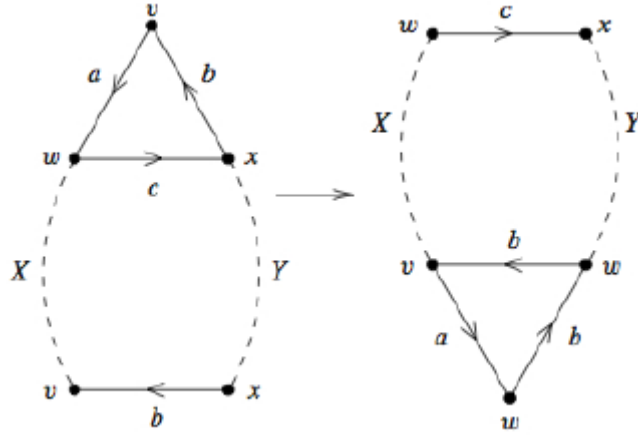
- (4) The surface M admits a presentation in which all vertices are identified to a single point. Recall that we have an equivalence relation on the set of edges based on identifications made, and this relation also forces an equivalence relation on the set of vertices. Choose some equivalence class $[v]$ of vertices and suppose that there is a vertex not in that equivalence class. Then there must be some edge a that connects a vertex in $[v]$ to a vertex in some other equivalence class $[w]$. The other edge that is incident with a at its vertex in $[v]$ can not be a^{-1} (if so, we would have eliminated it at step 2,) nor can it be a , or the quotient map would identify the initial and terminal vertices of a with each other, contradicting our assumption. We label this other edge b and the vertex at its opposite end x . Note that x may be part of $[v]$, $[w]$, or neither.

There must be another edge labeled either b or b^{-1} somewhere in the polygon. We may assume without loss of generality that it is b^{-1} ; the other proof is similar except for an extra reflection. Then the presentation is of the form $baXb^{-1}Y$, where X and Y are non-empty strings of edges. By elementary

transformations:

$$\begin{aligned}
\langle a, b, X, Y | baXb^{-1}Y \rangle &\cong \langle a, b, c, X, Y | bac, c^{-1}Xb^{-1}Y \rangle \\
&\cong \langle a, b, c, X, Y | acb, b^{-1}Yc^{-1}X \rangle \\
&\cong \langle a, c, X, Y | acYc^{-1}X \rangle.
\end{aligned}$$

Recall that the vertex from $[v]$ was the initial vertex of a and the terminal vertex of b , so by pasting together the edges labeled b , we have reduced the number of vertices in $[v]$ and increased the number of vertices in $[w]$. It is possible that we have introduced new adjacent complementary pairs; if so, repeat step 2 again. We repeat this process until we have eliminated $[v]$, and we continue to repeat this procedure until there is only one equivalence class of vertices.



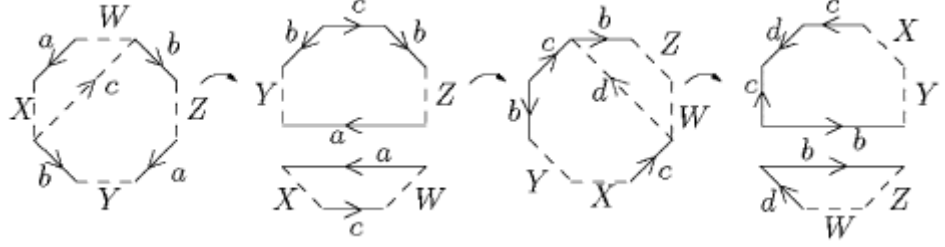
- (5) If the presentation has any complementary pair a, a^{-1} , then it has another complementary pair b, b^{-1} that occurs intertwined with the first, as in $a, \dots, b, \dots, a^{-1}, \dots, b^{-1}$. Assume that this is not the case. Then the presentation is of the form $aXa^{-1}Y$, where X and Y each contain only matched complementary pairs or adjacent twisted pairs. In other words,

each edge in X is identified only with another edge in X , and each edge in Y is identified only with another edge in Y . Recall that non-adjacent twisted pairs and adjacent complementary pairs are not possible at this point. Then the terminal vertices of a and a^{-1} are identified only with vertices in X , while the initial vertices are identified only with vertices in Y . This contradicts the fact that after step 4, all vertices identify to the same point.

- (6) There is a presentation of M in which all intertwined complementary pairs occur together with no other edges intervening. If this is not already the case, then the presentation is of the form $W a X b Y a^{-1} Z b^{-1}$, and we perform the following transformations:

$$\begin{aligned}
\langle a, b, W, X, Y, Z | W a X b Y a^{-1} Z b^{-1} \rangle &\cong \langle a, b, c, W, X, Y, Z | W a X c, c^{-1} b Y a^{-1} Z b^{-1} \rangle \\
&\cong \langle a, b, c, W, X, Y, Z | X c W a, a^{-1} Z b^{-1} c^{-1} b Y \rangle \\
&\cong \langle b, c, W, X, Y, Z | X c W Z b^{-1} c^{-1} b Y \rangle \\
&\cong \langle b, c, W, X, Y, Z | c^{-1} b Y X c W Z b^{-1} \rangle \\
&\cong \langle b, c, d, W, X, Y, Z | c^{-1} b Y X c d, d^{-1} W Z b^{-1} \rangle \\
&\cong \langle b, c, d, W, X, Y, Z | Y X c d c^{-1} b, b^{-1} d^{-1} W Z \rangle \\
&\cong \langle c, d, W, X, Y, Z | Y X c d c^{-1} d^{-1} W Z \rangle \\
&\cong \langle c, d, W, X, Y, Z | c d c^{-1} d^{-1} W Z Y X \rangle.
\end{aligned}$$

We can repeat this process for any other set of intertwined complementary pairs until none of them have other edges interrupting them.



- (7) Now M is homeomorphic to either a connected sum of one or more tori or a connected sum of one or more projective planes. All complementary pairs occur in intertwined groups $aba^{-1}b^{-1}$, which are tori, and all twisted pairs are adjacent to each other in pairs cc , which are projective planes. The presentation may consist exclusively of one type or the other, in which case the proof is complete. If it includes both types, then it must be of the form $aabcb^{-1}c^{-1}X$ or the form $bcb^{-1}c^{-1}aaX$. In either case, we refer to the remarks made before the proof showing that the connected sum of a torus and a projective plane is homeomorphic to the connected sum of three projective planes and repeat that process for all tori occurring in the presentation.

Examples:

$$\langle a, b, c | abacb^{-1}c^{-1} \rangle$$

We have a twisted pair a, a that is not adjacent. First, rotate the presentation so that it is in $VaWa$ form:

$$\langle a, b, c | cb^{-1}c^{-1}aba \rangle$$

Now cut along a new edge d :

$$\langle a, b, c, d | cb^{-1}c^{-1}ad, d^{-1}ba \rangle$$

Rotate:

$$\langle a, b, c, d | dcb^{-1}c^{-1}a, d^{-1}ba \rangle$$

Reflect the second word:

$$\langle a, b, c, d | dcb^{-1}c^{-1}a, a^{-1}b^{-1}d \rangle$$

Paste along a :

$$\langle b, c, d | dcb^{-1}c^{-1}b^{-1}d \rangle$$

Now we have eliminated a nonadjacent twisted pair, but we have another twisted pair b^{-1}, b^{-1} that is not adjacent. Repeat the above process by rotating again:

$$\langle b, c, d | ddc b^{-1}c^{-1}b^{-1} \rangle$$

Cut along a new edge e :

$$\langle b, c, d, e | ddc b^{-1}e, e^{-1}c^{-1}b^{-1} \rangle$$

Rotate the first word and reflect the second:

$$\langle b, c, d, e | eddcb^{-1}, bce \rangle$$

Paste along b :

$$\langle c, d, e | eddcce \rangle$$

One more rotation gives the final presentation as

$$\langle c, d, e | ddccee \rangle, \text{ which is the connected sum } \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2.$$

$$\langle a, b, c | abca^{-1}b^{-1}c^{-1} \rangle$$

Our first objective is to identify every vertex to the same point. Begin by cutting along a new edge d :

$$\langle a, b, c, d | abd, d^{-1}ca^{-1}b^{-1}c^{-1} \rangle.$$

Perform a few rotations on both words to obtain

$$\langle a, b, c, d | bda, a^{-1}b^{-1}c^{-1}d^{-1}c \rangle.$$

Paste together along a :

$$\langle b, c, d | bdb^{-1}c^{-1}d^{-1}c \rangle.$$

Now perform another rotation: $\langle b, c, d | cbdb^{-1}c^{-1}d^{-1} \rangle.$

In this form, it is obvious that cb and $b^{-1}c^{-1}$ can be consolidated to new symbols e and e^{-1} , respectively:

$$\langle d, e | ede^{-1}d^{-1} \rangle.$$

This is a standard presentation of \mathbb{T}^2 .

$$\langle a, b, c, d, e, f | abc, bde, c^{-1}df, e^{-1}fa \rangle$$

We need to reduce the presentation to one face. Paste the second and fourth words together along e :

$$\langle a, b, c, d, f | abc, c^{-1}df, bdfa \rangle.$$

Now paste the first and second words together along c and rotate the last word:

$$\langle a, b, d, f | abdf, abdf \rangle.$$

Reflect the second word to obtain

$$\langle a, b, d, f | abdf, f^{-1}d^{-1}b^{-1}a^{-1} \rangle.$$

Pasting together along f gives

$$\langle a, b, d | abdd^{-1}b^{-1}a^{-1} \rangle.$$

Folding along d and b in success to reveal that this presentation is nothing more than

$$\langle a | aa^{-1} \rangle,$$

a presentation of \mathbb{S}^2 .

We have now shown that all compact surfaces belong to one of the three classifications outlined. What remains is the possibility that members of any or all of these different classifications might be homeomorphic to one another by means of a clever homeomorphism that we have overlooked. The Klein bottle being homeomorphic to two projective planes was a surprise, as was the connected sum of a torus and projective plane being homeomorphic to the connected sum of three projective planes, so we may be remiss to ignore the possibility of other such results. To verify that the classes are indeed distinct, we will employ cohomotopy groups.

CHAPTER 5

HOMOTOPY AND COHOMOTOPY GROUPS

5.1. ALGEBRA PRELIMINARIES

DEFINITION 5.1.1. A group G is a structure consisting of a set of elements together with a map from $G \times G$ to G , also called an operation, that maps any two elements in G to a third element in G . The operation may be assigned a symbol such as $*$, and we would say that in G , $g_1 * g_2 = g_3$, with each $g_i \in G$. The following conditions must hold:

- (a) For all $a, b \in G$, $a * b \in G$ as well.
- (b) For all $a, b, c \in G$, $a * (b * c) = (a * b) * c$.
- (c) There exists some e (called the identity) in G such that $e * a = a * e = a$ for all $a \in G$.
- (d) For all $a \in G$, there exists some $b \in G$ such that $a * b = b * a = e$. The element b is called the inverse of a and can be denoted a^{-1} .

A group G with operation $*$ is called an *abelian* group if $a * b = b * a$ for all $a, b \in G$.

DEFINITION 5.1.2. Given two groups G and H , with operations $*$ and \odot , respectively, a group homomorphism from G to H is a function $h : G \rightarrow H$ such that for all x and y in G , $h(x * y) = h(x) \odot h(y)$.

This property of homomorphisms is often referred to as *preserving the operation*. If a group homomorphism exists between groups G and H , then G and H are said to be *homomorphic*.

DEFINITION 5.1.3. *Given two groups G and H , with operations $*$ and \odot , respectively, a group isomorphism from G to H is a bijective group homomorphism from G to H .*

If a group isomorphism exists between groups G and H , then G and H are said to be *isomorphic*. We write $G \cong H$.

5.2. HOMOTOPY AND COHOMOTOPY

DEFINITION 5.2.1. *Given topological spaces X and Y , let $C(X, Y)$ be the set of continuous maps $f : X \rightarrow Y$. Two maps f and $g \in C(X, Y)$ are homotopic if there exists $F \in C(X \times I, Y)$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$. If f and g are homotopic, we denote the relationship as $f \sim g$.*

LEMMA 5.2.2. *Homotopy is an equivalence relation on $C(X, Y)$.*

Proof. Let $f_1, f_2, f_3 \in C(X, Y)$. Obviously, if we let $F(t, x) = f_1(t)$ for all $x \in I$, then we have a homotopy from f_1 to itself, showing reflexivity. To check symmetry, assume that F is a homotopy between f_1 and f_2 . The function $G(x, t) = F(x, 1 - t)$ is a homotopy from f_2 back to f_1 . For transitivity, let us assume that $f_1 \sim f_2$ and $f_2 \sim f_3$. Then there exist homotopies F_1 (between f_1 and f_2) and F_2 (between f_2 and f_3 .) Define a new function $G : X \times I \rightarrow Y$ as follows:

$$G(x, t) = \begin{cases} F_1(x, 2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ F_2(x, 2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Essentially, G runs through the full continuous deformation from f_1 to f_2 between $t = 0$ and $t = \frac{1}{2}$, and then the same from f_2 to f_3 between $t = \frac{1}{2}$ and $t = 1$. Note

that G is well defined; $G(x, \frac{1}{2}) = F_1(x, 1) = f_2(x) = F_2(x, 0)$. Lemma 1.2.7 says that because G is, by definition, continuous on the two closed subsets $X \times [0, \frac{1}{2}]$ and $X \times [\frac{1}{2}, 1]$ of $X \times I$, it is continuous on all of $X \times I$. Therefore, G is a homotopy between f_1 and f_3 , and transitivity holds. Thus, \sim is indeed an equivalence relation.

The quotient set $C(X, Y)/\sim$ is denoted $[X, Y]$, and equivalence classes can be denoted by $[f]$, where f is some representative of the class.

PROPOSITION 5.2.3. *Let X_1 , X_2 , and X_3 be topological spaces. Let f_1 and g_1 be continuous maps in $[X_1, X_2]$ such that $f_1 \sim g_1$, and let f_2 and g_2 be continuous maps in $[X_2, X_3]$ such that $f_2 \sim g_2$. Then the compositions $[f_2 \circ f_1]$ and $[g_2 \circ g_1] \in [X_1, X_3]$ are also homotopic to each other.*

Proof. By definition, there exists a map $F_1 \in C(X_1 \times I, X_2)$ such that $F_1(x, 0) = f_1(x)$ and $F_1(x, 1) = g_1(x)$. Likewise, there is a map $F_2 \in C(X_2 \times I, X_3)$ such that $F_2(x, 0) = f_2(x)$ and $F_2(x, 1) = g_2(x)$. Define $\Delta : I \rightarrow I \times I$ via $\Delta(t) = (t, t)$, the diagonal embedding of the unit interval I in the square $I \times I$.

Applying the product of the identity function on X_1 and Δ , which is clearly continuous, to elements of $X_1 \times I$ results in an image in $X_1 \times I \times I$. We map from $X_1 \times I \times I$ to $X_2 \times I$ via the product of F_1 and the identity on I , also clearly continuous. Finally, we have F_2 , given to be continuous, that maps elements of $X_2 \times I$ to images in X_3 . Define $F_3 = F_2 \circ (F_1 \times \text{id}_I) \circ (\text{id}_{X_1} \times \Delta)$, a continuous map from $X_1 \times I$ to X_3 .

$$\begin{array}{ccc} X_1 \times I & \xrightarrow{F_3} & X_3 \\ \text{id}_{X_1} \times \Delta \downarrow & & \uparrow F_2 \\ X_1 \times I \times I & \xrightarrow{F_1 \times \text{id}_I} & X_2 \times I \end{array}$$

We can see the following:

$$\begin{aligned}
F_3(x, 0) &= F_2((F_1 \times \text{id}_I)(\text{id}_{X_1} \times \Delta)(x, 0)) \\
&= F_2((F_1 \times \text{id}_I)(x, 0, 0)) \\
&= F_2(f_1(x), 0) \\
&= f_2(f_1(x)), \text{ and} \\
F_3(x, 1) &= F_2((F_1 \times \text{id}_I)(\text{id}_{X_1} \times \Delta)(x, 1)) \\
&= F_2((F_1 \times \text{id}_I)(x, 1, 1)) \\
&= F_2(g_1(x), 1) \\
&= g_2(g_1(x)).
\end{aligned}$$

Therefore, F_3 is the required homotopy, and $(f_2 \circ f_1) \sim (g_2 \circ g_1)$.

Let $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ be the unit circle. Consider the function $\mu : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{S}^1$ defined by $\mu(z_1, z_2) = z_1 z_2$. Clearly, this function leads to an operation on \mathbb{S}^1 that is well-defined, associative, and commutative. We will use it to define a group structure on $C(X, \mathbb{S}^1)$ for any topological space X via $f * g = \mu \circ (f \times g)$.

PROPOSITION 5.2.4. *Let X be a topological space. The set $C(X, \mathbb{S}^1)$, when paired with the operation $*$ described above, forms a group structure.*

Proof. Let X be a topological space, consider the set $C(X, \mathbb{S}^1)$ of continuous functions from X to \mathbb{S}^1 , and define an operation $*$ on $C(X, \mathbb{S}^1)$ so that $f * g = \mu \circ (f \times g)$. Consider the quotient map from \mathbb{R} to \mathbb{S}^1 via $x \mapsto e^{2\pi i x}$. We can write any continuous map from X to \mathbb{S}^1 in the form $e^{2\pi i f(x)}$, where f is a suitable continuous map from X to \mathbb{R} . Using this approach, the group operation on $C(X, \mathbb{S}^1)$ becomes

$e^{2\pi i f(x)} e^{2\pi i g(x)} = e^{2\pi i (f(x)+g(x))}$, which clearly results in an abelian group.

PROPOSITION 5.2.5. *For any topological space X , $[X, \mathbb{S}^1]$ is an abelian group.*

Proof. We must show that the operation $*$ is well-defined modulo homotopy. The operation is commutative, so it suffices to show that if $f_2 \sim g_2$, then $f_1 \sim f_2 \sim f_1 * g_2$. Let $F \in C(X \times I, \mathbb{S}^1)$ such that $F(x, 0) = f_2(x)$ and $F(x, 1) = g_2(x)$. Consider the composition

$$X \times I \xrightarrow{\Delta \times \text{id}} X \times X \times I \xrightarrow{f_1 \times F} \mathbb{S}^1 \times \mathbb{S}^1 \xrightarrow{\mu} \mathbb{S}^1,$$

which we will define to be the function G . Then $G = \mu \circ (f_1 \times F) \circ (\Delta \times \text{id}) : X \times I \rightarrow \mathbb{S}^1$ is a continuous function such that $G(x, 0) = (f_1 * f_2)(x)$ and $G(x, 1) = (f_1 * g_2)(x)$. Therefore, $[f_1] * [f_2] = [f_1 * f_2]$ is a well-defined group operation on $[X, \mathbb{S}^1]$.

DEFINITION 5.2.6. *The group $[X, \mathbb{S}^1]$, as described, is called the first cohomotopy group of X . It is denoted $\pi^1(X)$.*

In line with our previous statements, $C(\mathbb{S}^1, \mathbb{S}^1)$ can be identified with the space of all functions of the form $e^{2\pi i f(x)}$, where f is a function from $[0, 1]$ to \mathbb{R} such that $f(1) - f(0)$ is an integer. Then we define the *degree map* $\deg : C(\mathbb{S}^1, \mathbb{S}^1) \rightarrow \mathbb{Z}$ via $\deg(f) = f(1) - f(0)$.

PROPOSITION 5.2.7. *The degree map defines an isomorphism $\pi^1(\mathbb{S}^1) \cong \mathbb{Z}$.*

Proof. Suppose that f and g are homotopic. Then a homotopy F between them is an element of $C(\mathbb{S}^1 \times [0, 1], \mathbb{S}^1)$, and thus it can be represented as $e^{2\pi i F(x)}$, where $F : ([0, 1] \times [0, 1]) \rightarrow \mathbb{R}$ such that $F(x, 0) = f(x) + k(x)$, $F(x, 1) = g(x) + h(x)$, and $F(1, t) = F(0, t)$ for all $(x, t) \in [0, 1] \times [0, 1]$. The functions h and k must be continuous integer-valued functions, and by the Intermediate Value Theorem, both must

be constant. Therefore, $G : [0, 1] \rightarrow \mathbb{R}$ via $G(t) = F(1, t) - F(0, t)$ is a continuous integer valued function such that $G(0) = \deg(f)$ and $G(1) = \deg(g)$. Again, by the Intermediate Value Theorem, G must be constant, so it follows that $\deg(f) = \deg(g)$, and the degree map is well-defined.

To see the \deg is surjective, take $n \in \mathbb{Z}$. Then the function $f(x) = nx$ has $f(0) = 0$ and $f(1) = n$, so $\deg(f) = n$. In order to see that \deg is injective, it suffices to verify that if $f(1) = f(0)$, then f is homotopic to a constant map. To define a homotopy, we first express each $(x, y) \in [0, 1] \times [0, 1]$ in the form (x, ax) for some $a \in \mathbb{R}$. Recall that 0 is identified with 1, so that $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$ are all the same point; otherwise, it is impossible to express $y \neq 0$ in the form ax when $x = 0$. For all $x \in [0, 1]$, we define $F(x, ax)$ as follows. If $a \in [0, 1]$, then we let $F(x, ax) = f(0) = f(1)$. If $a \in (1, \infty)$, then we let $F(x, ax) = axf\left(\frac{1}{a}\right) + (1 - ax)f(0)$. We have reached the important conclusion that $\pi^1(\mathbb{S}^1) \cong \mathbb{Z}$.

Let $\mathbb{D}^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$ be the closed unit disk in the complex plane.

PROPOSITION 5.2.8. *The first cohomotopy group of \mathbb{D}^2 is the trivial group.*

Proof. Let f be a member of $C(\mathbb{D}^2, \mathbb{S}^1)$. The boundary of \mathbb{D}^2 is simply \mathbb{S}^1 , so f restricted to its boundary is a continuous map from \mathbb{S}^1 to \mathbb{S}^1 . In order for f to extend to the interior of \mathbb{D}^2 , the restriction $f|_{\mathbb{S}^1}^1$ must be homotopic to a constant function. Thus, all such maps are homotopic to each other, and there is only one equivalence class. Since $\pi^1(\mathbb{D}^2)$ has only one element, it is the trivial group 0.

PROPOSITION 5.2.9. *The first cohomotopy group of \mathbb{S}^2 is the trivial group.*

Proof. The space \mathbb{S}^2 , as we have seen, can be constructed from \mathbb{D}^2 by identifying the boundary to a point. The space of functions $C(\mathbb{S}^2, \mathbb{S}^1)$ is the subspace of $C(\mathbb{C}, \mathbb{S}^1)$ of functions that are constant on $\mathbb{C} \setminus \mathbb{D}^2$. Therefore, $\pi^1(\mathbb{S}^2)$ is a subgroup of \mathbb{D}^2 , so

$\pi^1(\mathbb{S}^2)$ must also be the trivial group.

Let X_n be the quotient \mathbb{S}^1 / \sim_n , where \sim_n is the relation that identifies n distinct points of \mathbb{S}^1 to one point. For example, X_2 is a figure eight, and X_3 is three petals intersecting at one point.

PROPOSITION 5.2.10. *The first cohomotopy group of X_n is $\mathbb{Z} \oplus \mathbb{Z} \oplus \dots \oplus \mathbb{Z} = \mathbb{Z}^n$.*

Proof. First, we consider X_2 , the union of two copies of \mathbb{S}^1 with one point of intersection. Any map in $\pi^1(X_2)$ can be represented as the product $f \times g$, where f and g are both continuous maps from \mathbb{S}^1 to \mathbb{S}^1 . It follows that each such product map can be assigned to an ordered pair of integers $(n_1, n_2) \in \mathbb{Z} \oplus \mathbb{Z}$, where $n_1 = \deg(f)$ and $n_2 = \deg(g)$. The ordinary group operation is clear, and the same argument applies to X_n for any finite n .

PROPOSITION 5.2.11. *The first cohomotopy group of \mathbb{T}^2 is $\mathbb{Z} \oplus \mathbb{Z}$.*

Proof. This result follows directly from the previous proof and the fact that $\mathbb{T}^2 \cong \mathbb{S}^1 \times \mathbb{S}^1$.

PROPOSITION 5.2.12. *If Y_n is the connected sum of n tori, then $\pi^1(Y_n) \cong \mathbb{Z}^{2n}$.*

Proof. A connected sum of n tori, as we saw in Chapter 4, can be represented by a $4n$ -sided polygon with identifications on the boundary. In each, the vertices are mapped to a single point, which results in a space of the form X_{2n} . As we have seen, $\pi^1(X_{2n}) \cong \mathbb{Z}^{2n}$.

PROPOSITION 5.2.13. *The first cohomotopy group of \mathbb{P}^2 is \mathbb{Z}_2 .*

Proof. Observe that \mathbb{P}^2 is homeomorphic to \mathbb{D}^2 modulo the equivalence relation $(x, y) \sim (-x, -y)$ for (x, y) on the boundary of \mathbb{D}^2 . In the notation of Chapter 4, we may express it as a circle whose boundary is divided into two parts with identifications

aa. Considering the restriction of a function f to the boundary of this figure reveals that a loop traversing the boundary once must necessarily traverse it twice due to the identifications on the boundary. Therefore, a map f restricted to the boundary extends to the interior precisely if the degree of f is even. The first cohomotopy group contains two classes of elements, then, and must be isomorphic to \mathbb{Z}_2 .

Arguments of a similar nature show that $\pi^1(Z_n)$, where Z_n is the connected sum of n copies of \mathbb{P}^2 , is $\mathbb{Z}^{n-1} \oplus \mathbb{Z}_2$. For example, recalling that the Klein bottle is homeomorphic to the connected sum of two projective planes, $\pi^1(\mathbb{K}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$.

PROPOSITION 5.2.14. *If spaces X and Y are homeomorphic, then $\pi^1(X) \cong \pi^1(Y)$.*

Proof. Let f be the homeomorphism from X to Y . Then there exists a map $f^* : \pi^1(Y) \rightarrow \pi^1(X)$ via $[g] \mapsto [g \circ f]$, which is well-defined as a result of Proposition 5.2.3. Then f^* has $(f^{-1})^*$ as its inverse. Therefore, the two groups are isomorphic.

Since \mathbb{S}^2 , any connected sum of copies of \mathbb{T}^2 , and any connected sum of copies of \mathbb{P}^2 have first cohomotopy groups that are not isomorphic to one another, we can conclude that the spaces themselves are not homeomorphic to one another. The classifications that we have established are mutually exclusive.

BIOGRAPHY

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BIBLIOGRAPHY

- 1 J. M. Lee. *Introduction to Topological Manifolds*. Springer-Verlag, 2000.
- 2 B. Mohar and C. Thomassen. *Graphs on Surfaces*. Johns Hopkins University Press, 2001.
- 3 C. T. C. Wall. *A Geometric Introduction to Topology*. Addison-Wesley, 1972.